Mostly Complex – Splash at Berkeley

Nicholas Rui

March 2019

Contents

1 Paradigm

If you asked any random person off the street what they think math is about, they would probably venture to guess that math is the study of *numbers.* And, sure, math involves numbers, but I would argue that this is an incredible misrepresentation of the spirit of math.

Math isn't about numbers, or at least not *just* numbers. **Broadly, math cam be said to encompass the study of *patterns*, *rules*, and *logic*.** Whereas the sciences are bound to remain at least somewhat connected to reality, mathematicians are free to dream up whatever world they want. In one sense, mathematicians never say that anything *is true*. Rather, they ask *what if?* questions—*what if* there were infinitely extending lines following a certain set of rules (Euclidean geometry); *what if* there were things you could add together or subtract or scale up or down (linear algebra); *what if* there were some logic for describing things which are "close together" and "smooth" (analysis), or "containing" each other (set theory)? Does it matter if such things exist? Perhaps that's an interesting question, but not of primary concern to the mathematician.

Yet mathematicians also do not belong to the humanities. Despite not being bound to tangible things, there is still a certain objectivity to mathematics. Mathematicians are constrained by reason and consistency. Whereas they have the freedom to choose their starting point, they are guided by logic to immutable conclusions which set in stone right from the start. The mathematician asks *what if?*, and studies the *so what?*.

In this course, we will briefly outline one of the most useful *what if?* questions in the history of mathematics: *what if* there was some number that squared to 1.

2 Abstraction

When I was a kid, I would often think about math as describing objects that I could see or touch or visualize, or else relate to in some *visceral* way. The number 8 is not a thing that I could put into my backpack or give to a friend for their birthday, but the notion of 8 made sense to me—I could have 8 crayons or 8 notebooks or 8 candies. The notion of 1,584,482,293,572 also made sense to me as a natural extension of this, although I certainly would never have the patience to count that many things, and I certainly can't imagine that many of anything. Despite perfect triangles never existing in reality, I could still look around and see things that looked like triangles; I could still draw triangles.

Back then, at least for me, math was about what things *are*. Numbers *are* things we use to describe how many of something there are. Shapes *are* basically objects which we can draw on a paper. However, this approach to math is fundamentally limiting, both in terms of the preciseness of our language and the breadth of wonderful objects which we can explore.

The better approach is to **think about things in terms of what they do.** Numbers *add* and *multiply* together with certain rules like: it doesn't matter what order they add in. Shapes *follow* Euclid's five axioms from which you can, at least in principle, derive all of geometry without drawing a single line. Once you start thinking like this, you can start making up stuff that a thing might do, and then ask the all important question: if a thing did that stuff, what other stuff would it *have* to do?

3 Numbers

The first way in which we are introduced to numbers in school is by **counting things**. There could be 1 of something, or 2 of something, or 12 of something. And you could always add one more thing, and you should be able to describe that also. If you have two piles of things and combined them together, you should also be able to count the quantity of things within the composite pile. If you had a pile of piles of things, you should be able to count the number of things in the whole pile.

3.1 Naturals

We can take these rules and perform some **abstraction**; we extract rules from some specific examples and see what makes them tick. Specifically, if you had some *number* of things x, there is some action of adding another thing to the pile: σ. And we should get out another number y:

$$y = \sigma(x) \tag{1}$$

In addition, we should be able to combine piles together. We could pick some symbol to represent the act of combining piles together, say, “+.” If x and y are numbers, then we can also find another number z so that

$$z = x + y \tag{2}$$

You could also describe the act of getting together x piles of y things using the symbol “ ,” so that, for any two numbers x and y, we could also find some number w which obeys

$$w = x \quad y \tag{3}$$

Packaged together with some other rules which can be found many places, we have effectively described the **natural numbers** N, containing the numbers 1, 2, 3, and so on. So far, we shouldn't have much trouble imagining these. After all, we spend a large fraction of our early lives learning to count things.

3.2 Zero

While our system of numbers N can do a lot of things for us, there are some things that it can't do. For example, is there such a number e which, for *any* number x, obeys the following rule:

$$x + e = x \tag{4}$$

Maybe there *is*, and maybe there *isn't*. What does it even mean for a number to exist? But this question isn't germane. As mathematicians, we have the prerogative to just *declare* such a number to exist, and to clean up the consequences after. So let's pretend that there is such a number that plays by Rule 4.

Indeed, what if we had *two* such numbers, e and g, both of which followed Rule 4 and didn't affect the numbers that they added to. We know the number e can be rewritten $e + g$ since g doesn't change the number it's added to. But we also know that $e+g$ is the same as g since e *also* doesn't change the number that it's added to. Hence, e and g are the same number after all!

$$e = e + g = g \tag{5}$$

Since there's only one such number, we can just give a name, **zero**: 0. Now we have a special, privileged number, and, if we would like, we can add it to our collection of numbers.

3.3 Integers

We defined this notion of "adding" two numbers together. However, we haven't yet thought about going in reverse. Now we have a special, "privileged" number 0. Let's suppose we have some number x. Is there another number y which obeys the following rule?

$$x + y = 0 \tag{6}$$

Well, no. At least, not yet, but we can make it so. So we wave our wand and say that such a number y must exist. For a given x, how many y's are allowed? Suppose that there were two such numbers y and z that, when individually added to x, both gave 0. Following the same reasoning as before, we see that

$$y = 0 + y = (x + z) + y = (x + y) + z = 0 + z = z \tag{7}$$

Therefore, $y = z$; they are the *same* number. For every x, then, there is at most *one* number y that, when added to x, gives 0. Let's assume that such a number exists for every number, and give it a special name: $\quad x$. It's easy to see that $0 = \quad 0$, since $0 + 0 = 0$.

With these new additions, we have a whole lot of other numbers to work with. We now have the **integers** Z: 0, +1, +2, and so on, but also 1, 2, and so on.

Can we go even further? Sure! We can keep adding numbers to follow rules that we want followed until the cows come home. But before we do, we should appreciate that there is already incredible structure here. Even without extending our system of numbers even further, we can already understand questions asked by one particular branch of math: **number theory**. Indeed, when thinking about the integers, such numbers like $1/2$ or π or, later, $2 + 3i$ simply don't exist yet since we have never demanded that they do.

One can have a perfectly fruitful experience stopping here (or anywhere before). Yet we will not.

3.4 Rationals

We've added some extra numbers in order to enforce new rules extending the usefulness of the concept of addition. We can also try to do something like this for multiplication.

Specifically, if there is some number x and another number y, is there a number z such that

$$y \quad z = x \tag{8}$$

More informally, can we "undo" the concept of multiplication?

We see that, confined to the integers Z, we so far cannot. If you took $x = 3$ and $y = 2$ for example, there is no integer z such that

$$2 \quad z = 3 \tag{9}$$

But, again, what if we just declare that, for any x and y, there exists a number z which follows Rule 8? It turns out that this *almost* works. However, suppose $x = 0$. Is there such a number z which follows, say,

$$0 \quad z = 3 \tag{10}$$

We can't really define this number z in a consistent way, since the preexisting rules of our number system have $0 \quad z = 0$ for all numbers z. If we aren't to contradict any rules we already have, we have to amend our assertion by saying: Rule 8 is true for any numbers x and y *where* $y \neq 0$.

Then we obtain the **rationals Q**. These are just the fractional numbers that we are used to, including numbers we already had like 1 or 3 or 37 but also ratios of integers like 1/2 or 2/3 or 22/7.

3.5 Reals

When writing down rational numbers, we note that, if we adopt some kind of convenient system for writing rational numbers down like decimal, we can represent such numbers as possibly infinitely repeating decimal expansions. For example,

$$\begin{aligned} 1/2 &= 0.5 \\ 2/3 &= 0.\overline{6} \\ 22/7 &= \quad 3.\overline{142857} \end{aligned} \tag{11}$$

where a bar indicates that a sequence of numbers repeats forever. However, note that we could also imagine writing numbers which are sequences of possibly non-repeating digits. At the moment, those numbers would not count as "numbers," but, as mathematicians, we could just declare that they exist. These are called the **real numbers R**.

Why would such a concept even be useful? After all, any time we want to divide a cake among friends, we could just use some rational number; what purpose is there for numbers that go on and on without repetition so that, by definition, we could never write down the whole thing?

Well, note that, defining $x^2 = x \quad x$, there is some integer x, notably $x = 1$, such that

$$x^2 = 1 \tag{12}$$

In addition, note that we can also find a solution ($x = 2$) such that

$$x^2 = 4 \tag{13}$$

But can we find a rational number x such that

$$x^2 = 2 \tag{14}$$

Since all rational numbers are a fraction of some two integers such that the fraction is irreducible, say p and q so that the whole rational number is $x = p/q$, we can see if there is such a number that satisfies Equation 14. Notably, we enforce that the fraction is "irreducible," i.e., p and q cannot both be divided any further and still satisfy $x = p/q$. Substituting, we obtain

$$x^2 = p^2/q^2 = 2 \tag{15}$$

so that

$$p^2 = 2q^2 \tag{16}$$

However, 2 times any integer (such as q^2) is even, so $2q^2 = p^2$ is even. But if an integer squares to an even number, then it's also even, so p is even. If p is even, it can be written as 2 times another number, say $p = 2r$. But then we have

$$(2r)^2 = 4r^2 = 2q^2 \tag{17}$$

But then

$$q^2 = 2r^2 \tag{18}$$

By similar reasoning as before, we see that q must be even. But if both p and q are even, we should be able to divide both of them by 2 and get two integers with still divide to make x. Thus, the fraction is not reduced after all! Since our assumption that such a rational number x logically led to an inconsistency, this assumption must be flawed, and no such rational x can exist. This line of reasoning is called **proof by contradiction**.

Yet we can still find rational numbers which, in a sense, square to a number that is "close" to 2. For example,

$$(7/5)^2 = 1.4^2 = 1.96 \tag{19}$$

and we can get closer

$$(141/100)^2 = 1.41^2 = 1.98 \tag{20}$$

and closer

$$(707/500)^2 = 1.414^2 = 1.999396 \tag{21}$$

So, in a way, even though we can't write down every single digit of $\sqrt{2}$, by following rational numbers which *approach* the right answer, we can still figure out what the digits of $\sqrt{2}$ are. And using the real numbers means that we are allowed to *declare* that there is an x such that $x^2 = 2$.

But not all real numbers arise from algebraic expressions like above. Some numbers, so-called **transcendental numbers**, are real numbers which are not a "root" of any expression. It may seem like such numbers would be especially useless, but such numbers are famous throughout mathematics. Namely, numbers like the ratio of a circle's circumference to its diameter $\pi = 3.14159\ldots$ and Euler's (exponential) constant $e = 2.71828\ldots$ are two extremely useful transcendental numbers in mathematics.

4 Complex

Everything seems fine and dandy. We've basically reached the end of numbers that need to be understood before a student can graduate from a typical high school math curriculum. In this context, it's no surprise that complex numbers are a mystery to most people.

Recall that we were concerned before that $x^2 = 2$ didn't have an answer until we extended our number past the rational numbers. We simply declared such a number to exist and then saw, in hindsight, that this concept was useful. It doesn't really matter that you can't actually ever get $\sqrt{2}$ of anything; one still sees the utility in defining such a thing.

However, though the reals are quite expansive, there are still expressions we can write down which don't seem to have a real solution. For example, consider

$$x^2 = -25 \tag{22}$$

Clearly, there is no real number x which satisfies Equation 22. Any real number squared becomes a non-negative number; even squaring negative numbers just causes the minus sign to cancel out! So we're stuck.

But are we really? As mathematicians, we could just *declare* such a number x to exist and then worry about the consequences later. Surely, what's the harm in doing that?

Specifically, though, we could be more systematic about this. We could define some *new* number i which obeys the following rule:

$$i^2 = -1 \tag{23}$$

Many people make the mistake of saying that i *is* the square root of -1: $i = \sqrt{-1}$. While this rule of thumb is sometimes useful for simplifying expressions, it's ultimately not the best definition. For example, the number 1 has two square roots: 1 and -1. But, when we say that $i^2 = -1$, we are just enforcing some rule that i obeys. Surely there's nothing wrong with that.

Anyway, we see that, if such a number *were* to exist, we would be able to write down $x = 5i$ so that

$$(5i)^2 = 25i^2 = -25 \tag{24}$$

And we note that $x = -5i$ also does the job of satisfying Equation 22:

$$(-5i)^2 = 25i^2(-1)^2 = -25 \tag{25}$$

It's important to note that, even though we're using a letter i to denote this new number, i is *not* a variable. It's a whole new number, the **imaginary unit**, which we write with the letter i because there isn't a convenient representation in solely terms of digits we already have. Yet, despite being called "imaginary," i is no less real than 1. They're all just mathematical constructs, anyway.

But if we're used to adding two numbers together to get a third number (so-called **closure**), then we should also be able to add, say, 2 and i together to get $2 + i$. We could also multiply i to things to get other numbers like $2 + 3i$ or $7 + 4i$. Indeed, if we take two real numbers x and y and combine them in the following way,

$$z = x + iy \tag{26}$$

Then we have obtained the space of **complex numbers $\mathbb{C}$**. But, despite being called "complex," there's nothing particularly complex about them. I guess they're a composite of a real and imaginary part, so, in that sense, they're complex, but they're not particularly confusing. In a way, I guess one could say that they're *mostly complex*.

Note that, in a sense, complex numbers are *two-dimensional*. It takes two real numbers to specify a complex number. Whereas the real numbers can be plotted on a line, the complex numbers take up an entire *plane*, the **complex plane**.

4.1 Computation

Of course, given this new system of numbers, we can define very natural ways of adding and multiplying them together. Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we can add them together by just adding their real and imaginary components:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{27}$$

The real and imaginary parts of a complex number $z = x + iy$ are denoted by

$$\begin{aligned} \mathrm{Re}(z) &= x \\ \mathrm{Im}(z) &= y \end{aligned} \tag{28}$$

We can also use the FOIL method to figure out how complex numbers are supposed to multiply:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 \quad y_1y_2 \\ &= (x_1x_2 \quad y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned} \tag{29}$$

But there are some other operations of interest when it comes to complex numbers. For example, we can define something called the **complex conjugate**, which we write as a symbol:

$$z \ = x \quad iy \tag{30}$$

Essentially, the complex conjugate just switches all i's to $-i$'s which, on the complex plane, corresponds to a reflection across the x (real) axis.

Using the complex conjugate, we can generalize the **absolute value**. Recall that, for real numbers, the absolute value of a number just removes any minus signs, if any. In other words, $|2| = 2$ but $|-3| = 3$. In a sense, the absolute value measures the *distance* to the origin.

We can define a similar thing for complex numbers, the **norm** $|z|$ of a complex number z, which measures the distance from a complex number to the origin $0 + 0i$ on the complex plane:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{zz^*} \tag{31}$$

Finally, we can define the **argument** (alternatively "phase" or "angle") of a complex number is the counterclockwise angle from the real axis, specified to lie between $-\pi$ and π radians (or, alternatively, -180° to 180°):

$$\arg(z) = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) = \arctan\left(\frac{y}{x}\right) \tag{32}$$

All of these operations associated with a given complex number are shown on the drawing of the complex plane in Figure 1. From the drawing, we can make some interesting observations about a complex number z and its complex conjugate z^*:

1. z and z^* have the *same* real part: $\mathrm{Re}(z^*) = \mathrm{Re}(z)$.

2. z and z^* have *opposite* imaginary parts: $\mathrm{Im}(z^*) = -\mathrm{Im}(z)$.

3. z and z^* have the *same* norm: $|z^*| = |z|$.

4. z and z^* have *opposite* arguments: $\arg(z^*) = -\arg(z)$.

4.2 Repetition

Dealing with the real numbers, we can ask what happens when we multiply a number by itself over and over again. For example, for the number 2, we get:

$$2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32 \xrightarrow{\times 2} 64 \xrightarrow{\times 2} 128 \xrightarrow{\times 2} 256 \ldots \tag{33}$$

This is an operation of such importance that we call it **exponentiation**, with n^q meaning multiplying n by itself q times. In most cases, we can then extend the meaning of this operation for q being any real number.

However, note that, as we exponentiate to higher and higher powers, 2^q doesn't ever loop around. It just keeps getting bigger and bigger. Similarly, for a number like $1/2$, we have

$$\frac{1}{2} \xrightarrow{\times \frac{1}{2}} \frac{1}{4} \xrightarrow{\times \frac{1}{2}} \frac{1}{8} \xrightarrow{\times \frac{1}{2}} \frac{1}{16} \xrightarrow{\times \frac{1}{2}} \frac{1}{32} \xrightarrow{\times \frac{1}{2}} \frac{1}{64} \xrightarrow{\times \frac{1}{2}} \frac{1}{128} \xrightarrow{\times \frac{1}{2}} \frac{1}{256} \ldots \tag{34}$$

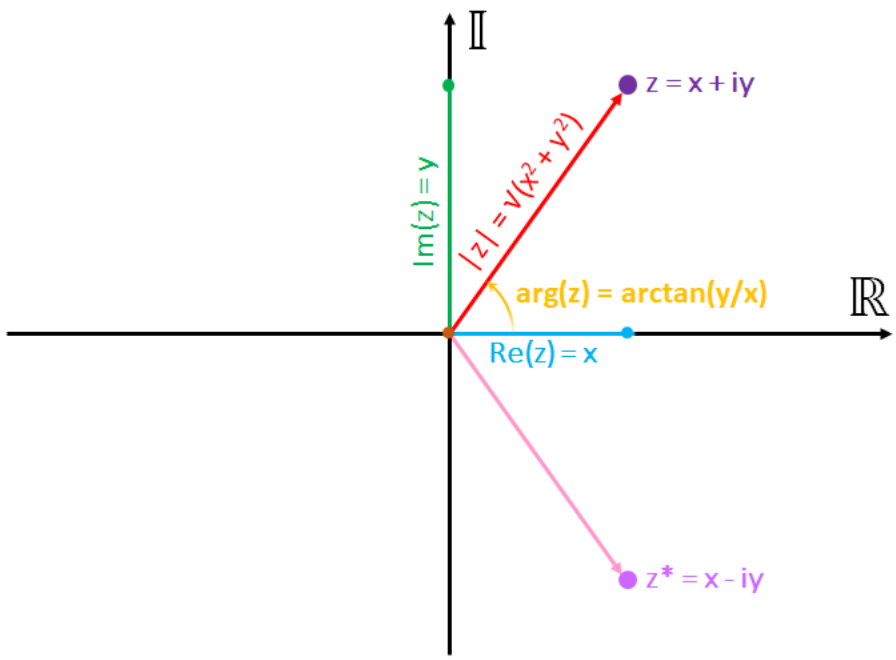

Figure 1: A schematic drawing of the complex plane showing some complex number $z = x + iy$ alongside its real and imaginary parts, norm, phase, and complex conjugate.

Now, $(1/2)^q$ gets *smaller* as q gets bigger. But it doesn't loop around.

Are there some choices for n such that, as you increase q, you loop around? Well, yes. One of the easiest examples is the number 0:

$$0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \ldots \tag{35}$$

Let's forget about this one, since 0 kills everything that it multiplies. But let's consider 1:

$$1 \xrightarrow{1} 1 \xrightarrow{1} 1 \xrightarrow{1} 1 \xrightarrow{1} 1 \xrightarrow{1} 1 \xrightarrow{1} 1 \xrightarrow{1} 1 \ldots \tag{36}$$

When exponentiated, 1 repeats on a period of one number. Every time you multiply 1 by itself, you'll get 1 again.

But 0 and 1 aren't the only numbers which behave this way. What about -1?

$$-1 \xrightarrow{-1} 1 \xrightarrow{-1} -1 \xrightarrow{-1} 1 \xrightarrow{-1} -1 \xrightarrow{-1} 1 \xrightarrow{-1} -1 \xrightarrow{-1} 1 \ldots \tag{37}$$

We see that -1, when "exponentiated," repeats on a period of 2. While multiplying a number by -1 doesn't change the number, multiplying by -1 *twice* does.

And if we were restricted to the real numbers, that would be the end of things. However, with the complex numbers, we have a number i which also has a "repetitive" property:

$$i \xrightarrow{i} -1 \xrightarrow{i} -i \xrightarrow{i} 1 \xrightarrow{i} i \xrightarrow{i} -1 \xrightarrow{i} -i \xrightarrow{i} 1 \ldots \tag{38}$$

We see that i, when exponentiated, repeats on a period of 4. And same with $-i$.

What if there were a number that repeated with a period of 3? In other words, is there some number such that, when I multiply it by itself 1 or 2 times, I don't get the original number back, but I do when I multiply it by itself 3 times? Of course, we're used to just declaring such numbers to exist, but, in this case, we should be careful. If we adopt the complex numbers, this number *already* exists. Notably, you can check that $z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ is such a number:

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \xrightarrow{z} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \xrightarrow{z} 1 \xrightarrow{z} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \xrightarrow{z} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \xrightarrow{z} 1 \xrightarrow{z} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \xrightarrow{z} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \ldots \tag{39}$$

But this is incredibly opaque. How did I even find this out? Surely I'm not some hermit night owl multiplying numbers together at 2 am seeing which convoluted numbers multiplied to themselves give the same number again (though this is not far from the truth).

4.3 Series

Let's take a quick aside to talk about the idea of a **Taylor series**. I won't explain the details here since doing so involves a bit of discussion about calculus, but we can motivate why we can do the amazing things outlined in this section.

First and foremost, consider a function like $\sin x$. Suppose that x is some really small number. Then, assuming that x is in units of radians, we can perform the following approximation:

$$\sin x \approx x \tag{40}$$

However, this might not be good enough. After all, sine wiggles around a lot, so, if our value of x is too big, we will not get the right answer, and we might not even get close! We might want to tack on an extra term to make this approximation work for longer.

In fact, the next term we need to add will look like this:

$$\sin x \approx x - \frac{x^3}{3!} \tag{41}$$

In fact, the exclamation mark after 3! doesn't mean that 3 is excited (though I would be if I got to participate in a Taylor expansion). Rather, it is shorthand for the **factorial**: $3! = 3 \times 2 \times 1 = 6$, which is where you multiply together all the whole numbers less than or equal to the number before the exclamation point. As another example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. The next term is

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{42}$$

However, we quickly get impatient, and wonder how many terms we need to add to get the *exact* answer. In fact, this requires an *infinite* number of terms (Figure 2), so that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{43}$$

Even though we have to consider an infinite number of terms, the form that those terms take follows a very simple pattern, namely that the sign alternates, x is raised to the next odd power from the previous one, and the denominator is the factorial of the same odd number.

Cosine has a very similar expansion as well, but with even terms instead of odd (Figure 3):

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{44}$$

There's another function that we can take the Taylor series of, and that's the exponential function, e^x. Although it doesn't repeat and blows up when x

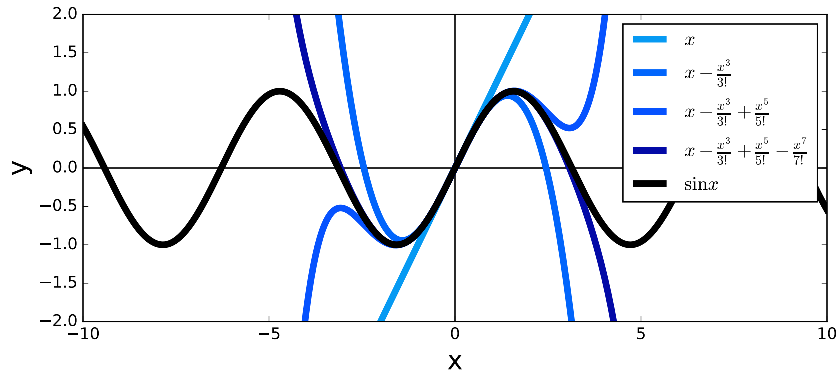

Figure 2: Successive Taylor series approximations for $\sin x$. Note that, the more terms we add, the closer the polynomial approximates $\sin x$. To get our approximation to be exact, we have to add an infinite number of terms.

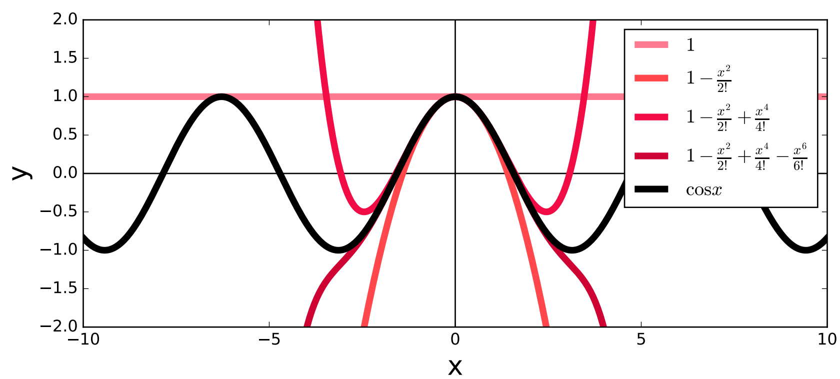

Figure 3: Successive Taylor series approximations for $\cos x$.

gets large, e^x still has a nice Taylor expansion (Figure 4):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{1} \frac{x^n}{n!} \tag{45}$$

We can see the pattern for this one as well. Actually, we notice that it's the same pattern as for the sine and cosine expansions, but with two differences: (1) we are now including *all* whole numbers and not just the odd or even ones, and (2) we are *not* alternating signs. So I guess they're not related after all.

But are we defeated? No!

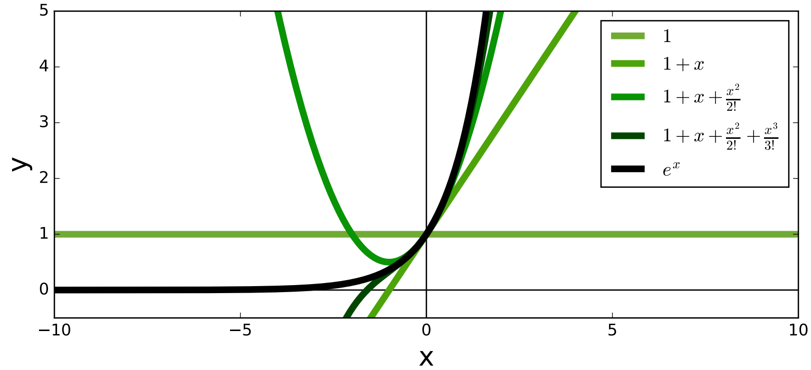

Figure 4: Successive Taylor series approximations for e^x.

4.4 Euler

We found the Taylor series for e^x, but, now, we're equipped with *complex numbers*. As mentioned before, complex numbers often have a strange property that, when you multiply them by themselves over and over, they loop around and you eventually get back to where you started. So what if I wanted to find the Taylor series of e^{ix}, an exponential with a pesky imaginary unit snuck in. In fact, finding this expansion is going to be as easy as taking the expansion for e^x and replacing x with ix everywhere. Then we will have

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \tag{46}$$

We can then simplify this, since we know what happens when we take i to the second or third or fourth power and so forth. We will find

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{47}$$

We see that, now, the signs of the terms alternate, but only every *two* terms. Also, every other term (the odd-powered ones) is now multiplied by an i. If we're clever, we notice that we can simply rearrange these terms to look like this:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{48}$$

But we *recognize* the grouped terms as the Taylor expansions for $\cos x$ and $\sin x$. Thus, we have determined a statement of fundamental importance in mathematics:

$$e^{ix} = \cos x + i \sin x \tag{49}$$

We have just derived the **Euler identity**, which reveals to us a stunning fact: the exponential of an imaginary number is *oscillatory*. If we take $x = \pi$ and do some rearranging, we get one of the most beautiful formulae in mathematics relating five of the most important numbers in mathematics together in one relation:

$$e^{i\pi} + 1 = 0 \tag{50}$$

In general, though, the full Euler identity is much more useful. Specifically, we notice that any complex number can be written not only in standard form but also in polar form:

$$z = x + iy = re^{i\theta} \tag{51}$$

where

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x) \end{aligned} \tag{52}$$

What kind of insight does this grant us? First off, the multiplication rule for complex numbers becomes simple, all of a sudden. Whereas standard form makes complex number addition very easy, so does polar form for complex number multiplication:

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{53}$$

We see that, if two complex numbers are represented as arrows in the complex plane, multiplying them together entails multiplying their norms and adding their phases.

The Euler identity also gives us a useful way to relate sines and cosines (which are often hard to work with) to exponentials (which are often easy to work with):

$$\begin{aligned} \cos x &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\ \sin x &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \end{aligned} \tag{54}$$

Indeed, the Euler identity often allows us to elide messy trigonometric identities in favor of addition. For example, suppose we had e^{x+y}. By the Euler identity, this is

$$e^{x+y} = \cos(x + y) + i\sin(x + y) \tag{55}$$

However, by the way exponentials work, we can also write

$$\begin{aligned} e^{x+y} &= e^x e^y \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y \\ e^{x+y} &= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y) \end{aligned} \tag{56}$$

But now we have two different ways of writing e^{x+y}, and we can set them equal:

$$\cos(x + y) + i\sin(x + y) = (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y) \tag{57}$$

For this to be true, the real and imaginary parts of both sides have to separately be equal. Thus, we have determined the double angle formulae for trigonometric ratios without even breaking a sweat:

$$
\begin{aligned}
\cos(x+y) &= \cos x \cos y \quad \sin x \sin y \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y
\end{aligned}
\tag{58}
$$

Was it obvious to you that the sine or cosine of the sum of two angles was some mess of sines and cosines? Me neither. But now, with the power of simple addition and the FOIL rule, you too can derive such peculiar rules without even stressing out your middle school self.

4.5 Rotation

What I've said is nice and all, but how do we *interpret* it? What kind of picture are we supposed to have in our head here? What even *is* a complex number? In fact, what even *is* a real number?

Is it meant to count something? That broke down when we started considering rational numbers. Is it meant to measure things? That went out the window when we generalized to real numbers, which have arbitrarily many (possibly non-repeating) digits that we could never possibly measure. So what are they?

I would submit that *one*[1] possible view is that numbers can be thought, at least in some abstract sense, as **transformations** that are applied via multiplication. Specifically, positive real numbers can be thought of **stretching** the number line by some amount, while negative numbers can be thought of **reflecting** the number line (across the origin) and *then* stretching.

In this sense, complex numbers can be thought of as **rotations** followed by stretching (Figure 5). We already saw that the complex numbers sort of have the structure of sines and cosines, so it is perhaps not surprising that they encode the structure of two-dimensional rotations. If the norm of a complex number is 1, then the complex number corresponds to a pure rotation and can be written via the Euler identity as $e^{i\theta}$ for some angle θ. Rather than having two such numbers, as with the real number line, we have an entire continuum of such numbers corresponding to the 360 of a circle.

In this context, i becomes very easy to interpret. We see that, when we multiply by i, a complex number $z = x + iy$ goes to $iz = \quad y + ix$ which, plotted on the complex plane, is at the location of the original complex number rotated counterclockwise about the origin by 90 . Obviously, two 90 rotations in sequence will flip your image around 180 corresponding to multiplying by 1.

Hence, thinking of complex numbers this way, we finally understand visually why $i^2 = \quad 1$ *isn't so weird after all.*

[1] of many

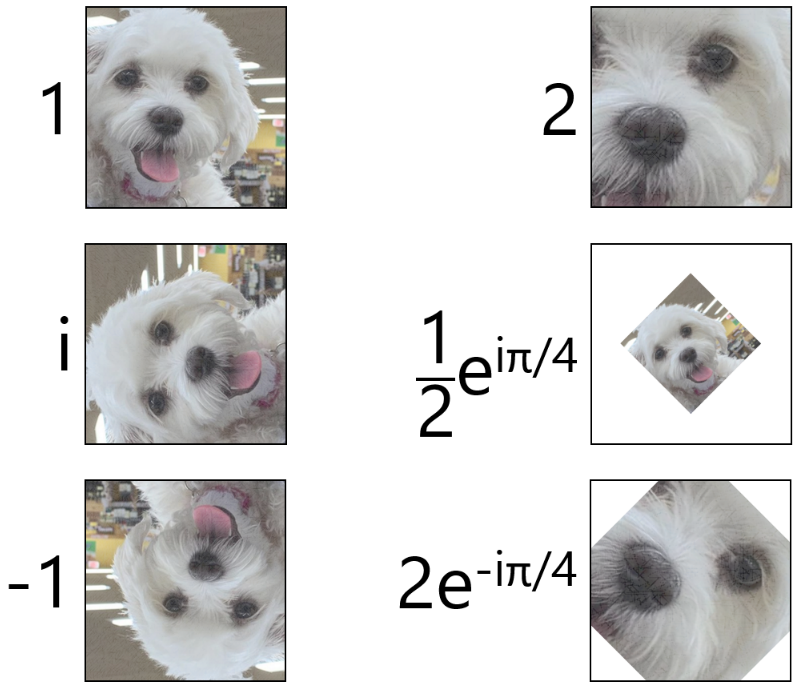

Figure 5: The effect of complex numbers on the complex plane with a picture of my dog mapped to it. The norm of the complex number determines how much the complex plane is scaled up and down, and the phase determines the angle by which the complex plane is rotated.

5 Evanescence

The math of complex numbers is already beautiful and needs no application (though their applicability to rotations probably already constitutes such a "practical" justification). However, complex numbers find extensive use in physics to describe a myriad of phenomena. In this section, we describe one of many such applications: the **evanescent wave**.

Since the nineteenth century, physicists have known that light is something called an **electromagnetic wave**, which is essentially just a traveling ripple in the everywhere-permeating electric and magnetic fields. We can write down the electric field of a traveling wave in one dimension as

$$E = E_0 \cos(kx \quad \omega t) \tag{59}$$

for some constant E_0 which is related to how bright the light is. However, like we said before, sines and cosines are sort of a pain to work with. What we might rather do is write E in terms of a complex exponential with the understanding that we will take the real part at the end. Hence, let's actually say that

$$\bar{E} = E_0 e^{i(kx \quad \omega t)} \tag{60}$$

In this expression, k and ω are called the **wavevector** and (angular) **frequency** and encode how long you must travel (for k) or wait (for ω) to get from one peak of the wave to the next. These constants k and ω are related to the velocity of the light v by

$$v = \frac{\omega}{k} \tag{61}$$

In a vacuum, when there's no other stuff around, light goes at the universal speed limit c. However, in a material with some index of refraction n[2] (which we usually think of as a real number at least 1), the speed of the light actually slows down to $v = c/n$. Then we can relate k and ω together so that we don't need to carry both around:

$$k = \frac{n\omega}{c} \tag{62}$$

Then our equation for the wave looks like this:

$$\bar{E} = E_0 e^{i\omega(nx/c \quad t)} \tag{63}$$

or, taking the real part,

$$E = E_0 \cos(\omega(nx/c \quad t)) \tag{64}$$

However, let's do something crazy and now replace n with $n \quad = n + i\kappa$ (where n and κ are real), a *complex* index of refraction. It's crazy, right? What does this even *mean*? Our electric field will look like

$$\bar{E} = E_0 e^{i\omega(n^* x/c \quad t)} = E_0 e^{i\omega((n+i\kappa)x/c \quad t)} = E_0 e^{\quad \omega\kappa x} e^{i\omega(nx/c \quad t)} \tag{65}$$

If we now take the real part, we get

$$E = E_0 e^{\quad \omega\kappa x} \cos(\omega(nx/c \quad t)) \tag{66}$$

This is almost the same answer as before with the important difference that this wave now *falls off with distance* (Figure 6). So, simply by taking n to be a complex number, we see that the imaginary part of n describes **absorption** of the light by a material. A wave which falls off like this is called an *evanescent wave*. Without doing much work, we have figured out a framework for describing what feels like an entirely *different* phenomenon from the speed of the wave.

[2] The index of refraction is related by Snell's law to how much light gets bent when passing from one material to another. It's why your straw appears bent when it's in a glass of water.

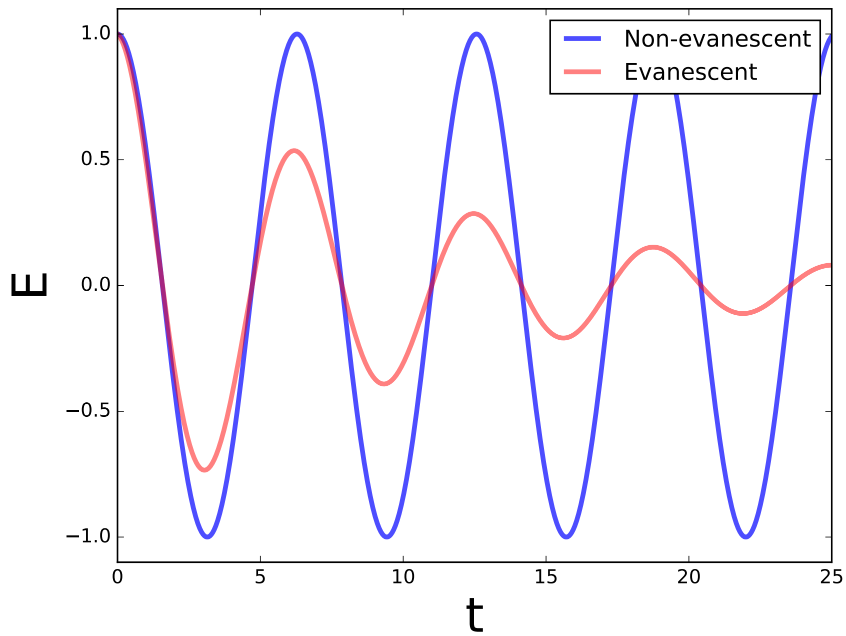

Figure 6: A non-evanescent and evanescent wave displayed side-by-side. The non-evanescent wave has a purely real index of refraction and so propagates forever whereas the evanescent wave has a complex index of refraction and so falls off with distance (which models the absorption of light by a material).

6 Generalizations

If there's one thing mathematicians like to do, it's generalize. In addition to asserting certain rules to be true, math practitioners frequency exercise an age-old penchant for *relaxing* rules, making them weaker so that they apply to more things, or tweaking rules slightly to see what happens. The complex numbers, in mathematical terms, belong to more general classes of objects, **manifolds** (curved spaces), **fields** (number-like systems), **vector spaces** (linear objects), **groups** (symmetries), **sets** (things containing other things), and many others.

6.1 Split-Complex

What if, instead of defining the imaginary unit i, we defined a new number j with the property that $j^2 = 1$ where $j \neq 1, -1$. You may be tempted to ask: why on earth would I want to do this? Surely there are already two real numbers that square to 1; why do we need to randomly decide that there's going to be a third and (since $(-j)^2 = 1$) fourth one?

Well, let's explore the consequences of this new, wacky space of **split-complex numbers**, numbers where, for any two real numbers x and y, can be written as

$$z = x + yj \tag{67}$$

Defining

$$z \quad = x \quad yj \tag{68}$$

We see that, now,

$$zz \quad = x^2 \quad y^2 \tag{69}$$

We see that curves of constant zz in the split-complex plane are no longer circles (as was the case for pedestrian complex numbers) but rather a hyperbola (Figure 7), with multiplication by split-complex numbers referring to the change in so-called **hyperbolic angles** (though restricted to one "leg" of the hyperbola).

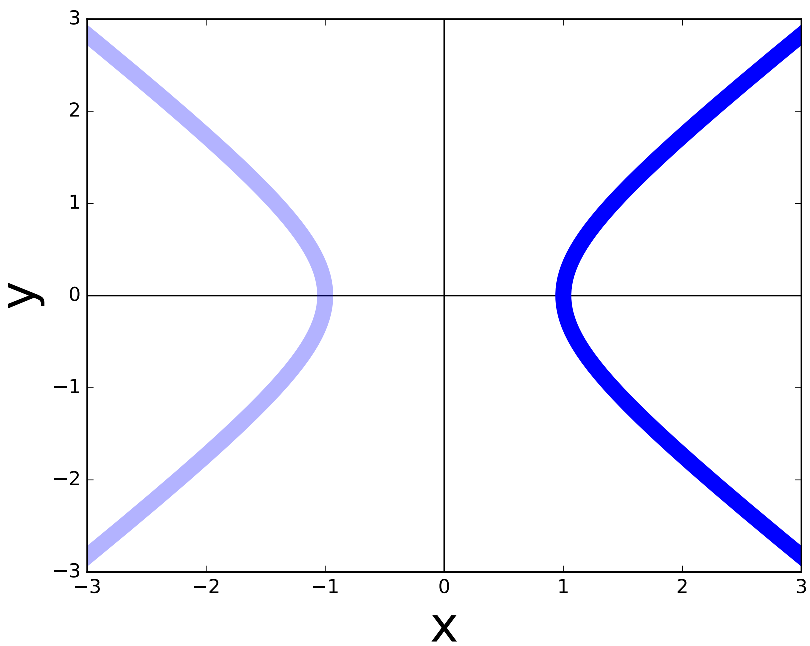

Figure 7: A unit hyperbola with one of the legs of the hyperbola emphasized. Just as complex numbers can be thought of as rotations, split-complex numbers can be thought of "hyperbolic rotations."

Indeed, making an analogy with Equations 54, we can define the so-called **hyperbolic trigonometric functions** (Figure 8),

$$\begin{aligned} \cosh x &= \frac{1}{2}\left(e^x + e^{\quad x}\right) \\ \sinh x &= \frac{1}{2}\left(e^x \quad e^{\quad x}\right) \end{aligned} \tag{70}$$

whereupon a very similar identity to the Euler identity can be proven:

$$e^{j\theta} = \cosh\theta + j\sinh\theta \tag{71}$$

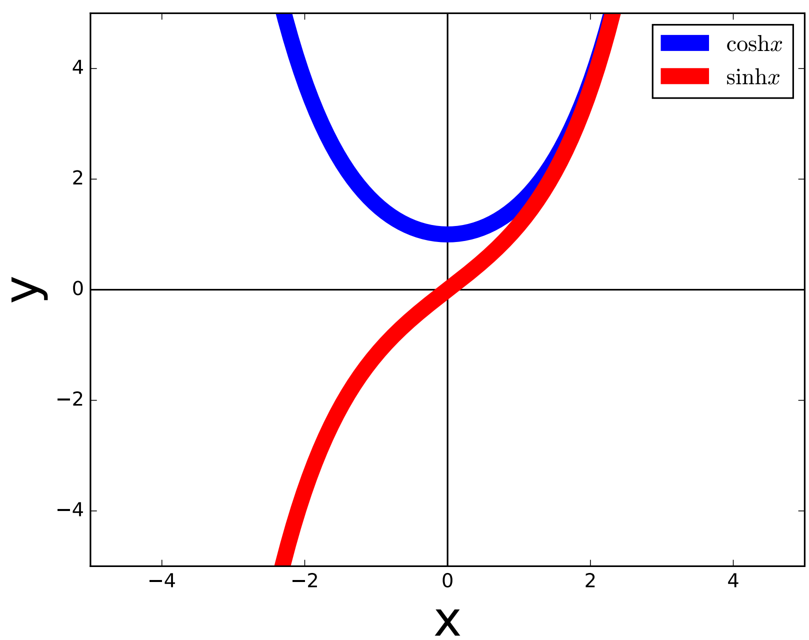

Figure 8: Hyperbolic trigonometric ratios $\cosh x$ and $\sinh x$ where x is the hyperbolic angle.

Well, that's nice, I suppose, but the more practical of you might be asking for an *application*. One need look no further than Einstein's theory of special relativity, which describes spacetime transformations between two reference frames moving relative to each other taking into account the finite speed of light. These so-called hyperbolic rotations exactly describe the changing of reference frames in special relativity, in which context they are called **Lorentz transformations**.

6.2 Dual

Suppose now we come up with a system of numbers with a new kind of number ϵ such that $\epsilon^2 = 0$ but ϵ isn't *itself* zero. This number system is called the **dual numbers**. Then the Euler formula becomes

$$e^{\epsilon x} = 1 + \epsilon x \tag{72}$$

Note that this looks just like the normal Euler formula but replacing sine and cosine with the first terms in their Taylor expansions (which is 1 for cosine and x for sine). It also looks like the first two terms of the Taylor expansion of e^x where x is replaced by ϵx. The dual numbers thus behave as if ϵ is a very small number so that $e^{\epsilon x}$ is very well-approximated by the first two terms of its Taylor series (though, using this definition of dual numbers, the Euler identity in Equation 72 is exact).

This is not particularly surprising once one notices that, if ϵ is thought to be some kind of "small" number, then ϵ^2 could be thought of as a number that is *so* small that it is basically zero. This is, of course, just interpretation after the fact; this identity follows as soon as we write down $\epsilon^2 = 0$.

6.3 Quaternions

Suppose we wanted to extend the complex numbers even further in order to describe higher dimensional rotations. The complex numbers are great and all, but they only describe rotations restricted to some two-dimensional plane. What if we wanted to rotate out of the page?

It turns out that we can make up a number system that describes three-dimensional rotations in the same way that the complex numbers describe two-dimensional rotations. Now, we have to define *three* new imaginary units i, j, and k following

$$i^2 = j^2 = k^2 = ijk = \quad 1 \tag{73}$$

This number system is called the **quaternions**, H, and quaternions will obey a slightly more complicated version of the Euler identity.

As cool as quaternions are, there is something very fundamentally different about them. If we think about two-dimensional rotations, applying two rotations will have the same effect regardless of which order you apply them in. However, three-dimensional rotations don't work that way. It's not hard to convince yourself that the final orientation of an object in three dimensions *very much* depends on the order of the rotations that you apply. Hence, *quaternions do not commute.* In fact, it can be shown that

$$\begin{aligned} ij &= \quad ji \\ ik &= \quad ki \\ jk &= kj \end{aligned} \tag{74}$$

Interestingly, when generalizing to the quaternions, we seem to have lost a lot of the structure we typically associate with numbers, like multiplication being **commutative**. While it's sort of weird to think about multiplication depending on the order of the things being multiplied, this property is physically motivated by three-dimensional rotations.

6.4 Beyond

In fact, we can (and people have) generalized even further. One (of many) directions that people have gone is extending the quaternions to the **octonions**

O in order to describe four-dimensional rotations, and even further to the **sedenions** S in order to describe five-dimensional rotations, and so on. Each time, the number of "unit" numbers you need will double: R has one (1), C has two (1, i), H has four (1, i, j, k), and, correspondingly, O and S have eight and sixteen, respectively.

But all is not quiet on the western front. It turns out that, each time you generalize up one dimension, you lose some mathematical structure that made your number system "nice." For example, when going from C to H, we lost commutativity. When going from H to O, it turns out that we also lose the **associativity** of multiplication. And when going to O to S, we lose a lesser known property called **alternativity** (the property that $x(xy) = (xx)y$ and $(yx)x = y(xx)$ for any numbers x and y). And, when it comes to this bleak picture of losing structure, it's turtles all the way down.

But what do we lose at the initial step, when we go from R to C? With all the gains we've discussed in this initial generalization, we should acknowledge one critical property that real numbers have which the complex numbers don't: **order**. Since the real numbers lie on a line, there is a very natural meaning to inequality symbols: $>$, $<$, , . It's very easy to talk about which number is *bigger* or *smaller* than another number. On a plane, we don't have this property anymore. And so, with this critical property in tow, the real numbers retain an uncompromised position in mathematics.