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Stellar Encounters and Dynamical Friction
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Outline *These Notes follow two of Subrahmanyan Chandrasekhar's seminal papers on stellar dynamics, namely* A Statistical Theory of Stellar Encounters *(Chandrasekhar 1941, [1]), which describes the details of a statistical treatment of stellar interactions in terms of a random walk in velocity space, and* Dynamical friction. I. General considerations: the coefficient of dynamical friction *(Chandrasekhar 1943, [2]), which introduces and quantifies the phenomenon of dynamical friction to resolve issues with the statistical treatments at times longer than a relaxation time. The Notes parallel the text quite closely and are not a succinct summary of the papers—they occasionally (for clarity) describe some steps more verbosely than in the aforementioned texts.*

Contents

1 *A Statistical Theory of Stellar Encounters*

In this Section, we summarize Chandrasekhar's classic work, *A Statistical Theory of Stellar Encounters*, [1], which constructs a statistical framework for describing the motion of a star due to its neighbors. Since gravity is a long-range force, it is not generally sufficient to model this motion as being due to a series of "two-body encounters" (impulsive "kicks" due to other stars that come within a certain range). Instead, a more statistical framework is needed to capture the effect of fluctuations in the distribution of the stellar field.

1.1 Description of a star's motion as a random walk

The gravitational force per unit mass (acceleration) which acts on a star due to a population of stars around it is given by

$$\vec{F} = -G \sum_n \frac{M_n}{|\vec{r}_n|^3} \vec{r}_n \tag{1}$$

respectively

where M_n and $\vec{r}_n$ are the masses and relative displacements of other stars within a certain range, respectively. Note that $\vec{F}$ will fluctuate in an unpredictable way, although we can hope to capture its statistical behavior. In particular, we define $W(F_x, F_y, F_z) = W(\vec{F})$ to be the probability density of $\vec{F}$, in particular, let

$$W(\vec{F})\, \mathrm{d}^3\vec{F} = W(F_x, F_y, F_z)\, \mathrm{d}F_x \mathrm{d}F_y \mathrm{d}F_z \tag{2}$$

be the probability of finding F_x between F_x and $F_x + \mathrm{d}F_x$, F_y between F_y and $F_y + \mathrm{d}F_y$, and F_z between F_z and $F_z + \mathrm{d}F_z$. Note that, if we have an isotropic (spherically symmetric) stellar distribution, we can convert this distribution to spherical coordinates and integrate over angles, i.e.,

$$W(\vec{F})\, \mathrm{d}^3\vec{F} = 4\pi F^2 W(\vec{F})\, \mathrm{d}F \equiv W(F)\, \mathrm{d}F \tag{3}$$

where $W(F)dF$ is the probability of now finding a force *magnitude* $|\vec{F}|$ between F and $F + \mathrm{d}F$. If we watch the stellar distribution for a long time, $W(F)\, \mathrm{d}F$ will be the fraction of that time where the total force on our star lies within this range.

However, to statistically predict the motion of our star, it is *not* sufficient to know the fraction of time that a given force intensity is applied. For concreteness, if the force fluctuates very quickly, the star will change directions much more frequently than it can move, and will likely not move far away from its starting point very quickly. In contrast, if the force fluctuates very slowly, the star will be able to move very far in a single direction before the force changes again. We see that we also need to know how long a given fluctuation lasts; in other words, we seek the "lifetime" of a given fluctuation. From studies of Brownian motion, the probability $\phi(t)\, \mathrm{d}t$ that a state decays between a time t and $t + \mathrm{d}t$ is[1]

$$\phi(t)\, \mathrm{d}t = e^{-t/T}\, \frac{\mathrm{d}t}{T} \tag{4}$$

[1] This generic exponential decay is due to the fact that a fluctuation cannot use its history to help determine the probability of whether it should decay at a given moment. The distribution ought to be self-similar if it is known at a certain time that the fluctuation has not already decayed.

where $T = T(F)$ can be called the lifetime of the state, and generally depends on F. While we do not know $T(F)$ at the moment, it is reasonable to guess that

$$T(\overline{F}) \sim \frac{\overline{D}}{\overline{v}} \tag{5}$$

where $\overline{F}$, $\overline{D}$, and $\overline{v}$ are the average forces, spatial separations, and velocities of the stars. The heuristic argument is that a statistical fluctuation with a typical force $\overline{F}$ will live about as long as a given configuration of stars is maintained—since the stars in a system will have had enough time to move a distance comparable to their separations in a time $\sim \overline{D}/\overline{v}$ (i.e., the system will have "rearranged" "significantly"), this time should be comparable to $T(\overline{F})$.

Then a given fluctuation modifies the velocity of our star by $\Delta\vec{v}$ given by

$$\Delta\vec{v} = T(F)\vec{F} \tag{6}$$

Then the velocity of our star after a long period of time will be given by an accumulation of such kicks, i.e.,

$$\sum_i \Delta\vec{v}_i = \sum_i T(F)\vec{F} \tag{7}$$

which describes the motion of our star as a random walk. **Note that understanding the specifics of this random walk requires understanding the distribution of force fluctuations W(F) and the fluctuation lifetime T(F).**

1.2 Finding $W(F)$ considering all other stars

We defined $W(F)$ as being the distribution of gravitational forces due to a sum of randomly distributed point sources with a number density N. This is mathematically analogous to the same quantity for randomly distributed electric charges, which is relevant to plasma physics and has been solved before by Holtsmark (see [3, 4]) as (translated to the gravitational problem)

$$W(F) = \frac{2F}{\pi} \int_0^\infty \exp\left(-\frac{8}{15}\sqrt{2\pi}\pi(GM)^{3/2}N\rho^{3/2}\right)\rho\sin(F\rho)\,\mathrm{d}\rho \tag{8}$$

which is the *Holtsmark distribution.*

We can define a characteristic force per unit mass for this problem,

$$Q_H \equiv \left(\frac{8}{15}\sqrt{2}\right)^{2/3} \pi GMN^{2/3} \approx 2.603GMN^{2/3} \tag{9}$$

whereupon Equation 8 becomes

$$W(F) = \frac{2}{\pi F} \int_0^\infty \exp\left(-\frac{Q_H}{F}x\right) x\sin x\,\mathrm{d}x \tag{10}$$

It can be shown that, for large F,

$$W(F) \to \frac{3}{2} \frac{Q_H^{3/2}}{F^{5/2}} \tag{11}$$

which reflects a relatively "slow" decrease of $W(F)$ with force (forces with large fluctuations are unphysically probable). The mean force squared $(\overline{F^2})$ computed using this $W(F)$ diverges, implying that $W(F)$ requires modification at large F.

1.3 Finding $W(F)$ considering only the nearest neighbor

It is reasonable to guess that the nearest neighbor has a dominating influence on the force felt by our star. Under this assumption, we can define $w(r)\,\mathrm{d}r$ to be the probability that the nearest neighbor lies at a distance between r and $r + \mathrm{d}r$ away from our star. Note that this is *not just* the probability that any star lies within this range, but specifically that the *nearest neighbor* lies within this range.

The probability that a star (not necessarily the nearest) lies within r and $r + \mathrm{d}r$ of our star is $(4\pi r^2\,\mathrm{d}r)N$ (the shell volume multiplied by the number density). The probability that a star lies within this range *and* there is a closer star is $4\pi r^2 N \int_0^r w(r')\,\mathrm{d}r'$. Wishing to exclude the latter possibility, we can subtract the two and find that

$$w(r) = \left(1 - \int_0^r w(r')\,\mathrm{d}r'\right) 4\pi r^2 N \tag{12}$$

Taking a derivative of this, we can find that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{w(r)}{4\pi r^2 N}\right) = -w(r) = -4\pi r^2 N \left(\frac{w(r)}{4\pi r^2 N}\right) \tag{13}$$

We notice that Equation 13 is an exponential equation in $w(r)/4\pi r^2 N$, and is accordingly solved by

$$w(r) = 4\pi r^2 N \exp\left(-\frac{4\pi r^3 N}{3}\right) \tag{14}$$

This expression for $w(r)$ comports with our expectations that $w(r) \approx 4\pi r^2 N$ when r is very small (since the probability of there being an *even closer* star is vanishing small).

The force due to this nearest neighbor is simply $F = GM/r^2$. Assuming that the nearest neighbor accounts for the entire force, we see that

$$W(F)\,\mathrm{d}F = w(r)\,\mathrm{d}r = w(r(F)) \left|\frac{\mathrm{d}r}{\mathrm{d}F}\right| \mathrm{d}F = \exp\left(-\frac{4\pi (GM)^{3/2} N}{3F^{3/2}}\right) \times 2\pi N (GM)^{3/2} \frac{\mathrm{d}F}{F^{5/2}} \tag{15}$$

Then we can define a characteristic force

$$Q \equiv \left(\frac{4}{3}\pi\right)^{2/3} GMN^{2/3} \approx 2.599 GMN^{2/3} \approx Q_H \tag{16}$$

whereupon

$$W(F)\,\mathrm{d}F = \frac{3}{2}Q^{3/2}e^{-(Q/F)^{3/2}}\frac{\mathrm{d}F}{F^{5/2}} \tag{17}$$

Note that, when $F \to \infty$, this becomes

$$W(F) \to \frac{3}{2}\frac{Q^{3/2}}{F^{5/2}} \tag{18}$$

capturing nearly the identical (problematic) behavior that was found in Equation 11 for large F. Chandrasekhar claims that the distributions in Equations 10 and 15 derived in these two ways behave practically identically except at extremely low F (which are negligibly unimportant in the statistical theory).

Moreover, the fact that we can retrieve the problematically large $W(F)$ at large F using only the nearest neighbor indicates that the nearest neighbor is responsible for this unphysically large force.

1.4 Modifying $W(F)$ at large F: our star affects its surroundings

The unphysically frequent large force fluctuation is due to the fact that we have erroneously assumed that the stellar distribution is uniform (with number density N) even very close to the neighborhood of our star. In reality, we ought to expect that our star should affect the probability that other stars are very nearby. In particular, if another star has speed v and is not bound to our star (i.e., in a binary), then that star may not come closer in distance than $r = r(v)$ given by a conservation of energy argument:

$$\frac{1}{2}v^2 = \frac{GM}{r(v)} \tag{19}$$

where M is the mass of our star, so that

$$r(v) = \frac{2GM}{v^2} \tag{20}$$

Since this sets a lower bound on how close another star can be, we see that this qualitatively resolves our problem.

While a full phase space treatment is appropriate, Chandrasekhar provides a simpler treatment in this particular work, which we reproduce. Using this knowledge, we can again attempt to compute $w(r)$ (the probability distribution of distance to the nearest neighbor),

$$w(r) = \left(1 - \int_0^r w(r')\,\mathrm{d}r'\right) 4\pi r^2 \chi(r) N \tag{21}$$

which is analogous to Equation 12, but where we have included a function $\chi(r)$ as an "adjustment" to account for the fact that stars may not come too close. Since we would like to modify $w(r)$ only at small r, it is reasonable to take

$$\begin{aligned} \lim_{r\to 0} \chi(r) &= 0 \\ \lim_{r\to\infty} \chi(r) &= 1 \end{aligned} \tag{22}$$

Then, as before, we may differentiate Equation 21 to find

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{w(r)}{4\pi r^2\chi(r)N}\right) = -w(r) = -4\pi r^2\chi(r)N\left(\frac{w(r)}{4\pi r^2\chi(r)N}\right) \tag{23}$$

which is now solved by

$$\begin{aligned} w(r) &= \exp\left(-4\pi N\int_0^r r'^2\chi(r')\,\mathrm{d}r'\right)\times 4\pi r^2\chi(r)N \\ &= \exp\left(-4\pi r^3\overline{\chi}N/3\right)\times 4\pi r^2\chi(r)N \end{aligned} \tag{24}$$

where for simplicity we have defined

$$\overline{\chi} = \frac{3}{r^3}\int_0^r r'^2\chi(r')\,\mathrm{d}r' \tag{25}$$

where we can see that

$$\begin{aligned} \lim_{r\to 0}\overline{\chi}(r) &= 0 \\ \lim_{r\to\infty}\overline{\chi}(r) &= 1 \end{aligned} \tag{26}$$

As a very rough approximation, let us assume that, for a given v, stars *are* uniformly distributed in space when $r > 2GM/v^2$, and that they are nonexistent within $r < 2GM/v^2$. Then we can write down $\chi(r)$, integrating this condition over all possible velocities,

$$\chi(r) = \iiint_{|\vec{v}|=\sqrt{2GM/r}}^{|\vec{v}|=\infty} f(\vec{v})\,\mathrm{d}^3\vec{v} \tag{27}$$

where $f(\vec{v})$ is the stellar velocity distribution. For concreteness, if the velocity distribution is Maxwellian, i.e.,

$$f(\vec{v}) = \frac{j^3}{\pi^{3/2}}e^{-j^2|\vec{v}|^2} \tag{28}$$

then

$$\chi(r) = \frac{4j^3}{\pi^{1/2}}\int_{\sqrt{2GM/r}}^{\infty} e^{-j^2v^2}v^2\,\mathrm{d}v = \frac{4}{\pi^{1/2}}\int_{a/\sqrt{r}}^{\infty} e^{-y^2}y^2\,\mathrm{d}y \tag{29}$$

$y \equiv jv$ and $a \equiv \sqrt{2GM}j$. Chandrasekhar then solves numerically for $\chi(r)$ and $\overline{\chi}(r)$ as a function of $a/\sqrt{r}$ and finds empirically that one may approximate

$$\chi(r) = \begin{cases} 1 & a \le r^{1/2} \\ 0 & a > r^{1/2} \end{cases} \tag{30}$$

Then, using Equation 24, we have

$$w(r) = \begin{cases} \exp\left(-4\pi N(r^3 - r_0^3)/3\right)\times 4\pi r^3 N & a \le r^{1/2} \\ 0 & a > r^{1/2} \end{cases} \tag{31}$$

where $r_0 \equiv 2GMj^2$. Then, once again attributing the force felt by our star purely to its nearest neighbor, Equation 31 implies that

$$W(F) = \frac{3}{2}Q^{3/2} \exp\left(-Q^{3/2}\overline{\chi}\left(\sqrt{GM/F}\right)/F^{3/2}\right) \frac{\chi\left(\sqrt{GM/F}\right)}{F^{5/2}} \tag{32}$$

Applying the approximation of Equation 30 reveals the imposition of a roughly hard cutoff of $W(F)$ at high F which resolves the issue of unphysically probable large force fluctuations:

$$W(F) = \begin{cases} \frac{3}{2}Q^{3/2} \exp\left(-Q^{3/2}\left(F^{-3/2} - F_{\max}^{-3/2}\right)\right) F^{-5/2} & F \leq F_{\max} \\ 0 & F > F_{\max} \end{cases} \tag{33}$$

where $F_{\max} \equiv 1/4GMj^4$.

1.5 Incorporating the large-F modification into the Holtsmark distribution

In our crude model, we have assumed that stars are distributed purely uniformly subject to the constraint that they may not lie within a characteristic distance $r = 2GM/v^2$ of each other. For a fixed v, this is equivalent to the Coulombic problem of enforcing a finite extent for ions interacting electrically, which was solved by Gans [5] and Holtsmark [4] for the same problem:

$$W(F) = \frac{2F}{\pi} e^{4\pi r_0^3 N/3} \int_0^\infty \exp\left(-(Q_H \rho)^{3/2}\right) K(\rho) \rho \sin(F\rho)\, \mathrm{d}\rho \tag{34}$$

where $K(\rho)$ is defined in [4].

1.6 Finding the fluctuation lifetime $T(F)$

In the limit where only the nearest neighbor contributes to the force fluctuation, we may apply Smoluchowski's result from the study of Brownian motion [6, 7, 8, 9] to obtain a rough approximation of $T(F)$. In particular, the lifetime of a state in which n particles are confined to a volume element σ is

$$T = \frac{\sqrt{6\pi}}{\sqrt{\overline{v^2}(n+\nu)}} \frac{\sigma}{S_\sigma} \tag{35}$$

where S_σ is the surface area of the volume element and $\nu \equiv N\sigma$ is the average number of particles expected at the mean density N. Using $\sigma = 4\pi r^3/3$, $S_\sigma = 4\pi r^2$, $n = 1$, and $\nu = 4\pi r^3 N/3$ (seeking the expected time that a given star is the sole occupant of a sphere of radius r), we have

$$T(r) = \sqrt{\frac{2\pi}{3\overline{v^2}}} \frac{r}{\frac{4}{3}\pi r^3 N + 1} \tag{36}$$

or, using $r = \sqrt{GM/F}$,

$$T(F) = \sqrt{\frac{2\pi GM}{3\overline{v^2}}} \frac{F}{Q^{3/2} + F^{3/2}} \tag{37}$$

Chandrasekhar argues that this approximation is likely acceptable for most purposes.

1.7 The behavior of a star's random walk

As a very simple first-pass model, consider an initially stationary star which, at regular time intervals, experiences a kick TF in a random, uncorrelated direction. After s intervals, the probability that the star has a speed mFT follows the usual random walk expectation:

$$P_m = \sqrt{\frac{3}{2\pi s}} e^{-3m^2/2s} \tag{38}$$

when s is large ($P_m = P_m \Delta m$ where $\Delta m = 1$). Since roughly $\Delta v = mFT$ and $t = sT$ in our problem, we have

$$P(\Delta v)\, \mathrm{d}(\Delta v) = \sqrt{\frac{3}{2\pi F^2 T t}} e^{-3|\Delta v|^2/2F^2Tt}\, \mathrm{d}(\Delta v) \tag{39}$$

so that the mean squared velocity of the distribution can be read off as

$$\overline{v^2} = F^2 T t \tag{40}$$

We can generalize Equation 39 to the case of varying F and $T(F)$ by applying the addition theorem for Gaussian errors, yielding

$$P(\Delta v)\, d(\Delta v) = \sqrt{\frac{3}{2\pi \overline{F^2 T} t}} e^{-3|\Delta v|^2/2\overline{F^2 T} t}\, \mathrm{d}(\Delta v) \tag{41}$$

where

$$\overline{F^2 T} = \int_0^\infty W(F) F^2 T(F) dF \tag{42}$$

and now

$$\overline{\Delta v^2} = \overline{F^2 T} t \tag{43}$$

Combining Equations 32 (for $W(F)$) and 37 (for $T(F)$) with Equation 42 yields

$$\overline{F^2 T} = \frac{3}{2}\sqrt{\frac{2\pi G M}{3\overline{v^2}}} Q^{3/2} \int_0^\infty \frac{F^{1/2}}{Q^{3/2} + F^{3/2}} \exp\left(-Q^{3/2}\overline{\chi}\left(\sqrt{GM/F}\right)/F^{3/2}\right) \chi\left(\sqrt{GM/F}\right) \mathrm{d}F \tag{44}$$

Defining $x \equiv Q^{3/2}/F^{3/2}$, we have

$$\overline{F^2 T} = 2\left(\frac{2\pi}{3}\right)^{3/2} \frac{G^2 M^2 N}{\sqrt{\overline{v^2}}} \int_0^\infty \exp\left(-x\overline{\chi}\left(\sqrt{GM/Q} x^{1/3}\right)\right)\left(\frac{1}{x} - \frac{1}{x+1}\right) \chi\left(\sqrt{GM/Q} x^{1/3}\right) \mathrm{d}x \tag{45}$$

Using Equation 16, we note that

$$\sqrt{\frac{GM}{Q}} x^{1/3} = \left(\frac{x}{\frac{4}{3}\pi N}\right)^{1/3} = D x^{1/3} \tag{46}$$

where $D^{-1} \equiv (4\pi N/3)^{1/3}$ is an inverse characteristic separation distance, and thus

$$\frac{a}{\sqrt{r}} = \sqrt{\frac{2GMj^2}{D}} x^{-1/6} \tag{47}$$

We may write the following scaling relation:

$$\frac{D}{2GM} = 2.33 \times 10^4 \left(\frac{D}{1\,\mathrm{pc}}\right) \left(\frac{M}{M_\odot}\right)^{-1} \left(10\,\mathrm{km\,s^{-1}}\right)^{-2} \tag{48}$$

We see that characteristically $a^2/D = 2GMj^2/D \sim 10^{-4}$, so that χ and $\overline{\chi}$ (which take as an argument the expression in Equation 46) only differ appreciably from unity when $x < 10^{-11}$. Taking $\chi = \overline{\chi} = 1$ whenever not multiplied by a function which diverges at $x = 0$, the integral in Equation 45 becomes

$$\begin{aligned} &\int_0^\infty \exp\left(-x\overline{\chi}\left(\sqrt{GM/Q}x^{1/3}\right)\right)\left(\frac{1}{x} - \frac{1}{x+1}\right)\chi\left(\sqrt{GM/Q}x^{1/3}\right)\,\mathrm{d}x \\ &= \int_0^\infty \frac{e^{-x}}{x}\chi(Dx^{1/3})\,\mathrm{d}x - \int_0^\infty \frac{e^{-x}}{x+1}\,\mathrm{d}x \equiv J \end{aligned} \tag{49}$$

Defining

$$-E(-x) = \int_x^\infty \frac{e^{-x}}{x}\,\mathrm{d}x \tag{50}$$

and using Equation 29 (for $\chi(r)$), we have, after a messy integration by parts,

$$J = -\frac{4}{\pi^{1/2}} \int_0^\infty E\left(-\left[\frac{2GMj^2}{D}\right]^3 z^{-6}\right) e^{-z^2} z^2\,\mathrm{d}z - 0.5963 \tag{51}$$

Once again note that $2GMj^2/D$ is incredibly small in realistic systems (Equation 48), so we may use the asymptotic expansion

$$E(-x) = \ln x + 0.5772 + \mathcal{O}(x) \tag{52}$$

which implies that

$$J = 3\ln\left(\frac{D}{2GMj^2}\right) - 0.5772 + \frac{6}{\pi^{1/2}} \int_0^\infty e^{-x} x^{1/2} \ln x\,\mathrm{d}x - 0.5963 \tag{53}$$

Within this expression appears the integral

$$\int_0^\infty e^{-x} x^{1/2} \ln x\,\mathrm{d}x = \Gamma\left(\frac{3}{2}\right)\left[\frac{\mathrm{d}\ln\Gamma(x)}{\mathrm{d}x}\right]_{x=3/2} = \frac{\pi^{1/2}}{2} \times 0.03649 \tag{54}$$

Therefore, finally,

$$J = 3\ln\left(\frac{D}{2GMj^2}\right) - 1.0640 \tag{55}$$

Finally, substituting Equation 55 (for J) into Equation 45 (for $\overline{F^2T}$) and then into Equation 43 (for $\overline{\Delta v^2}$) we have

$$\overline{\Delta v^2} = 6\left(\frac{2\pi}{3}\right)^{3/2} \frac{GM^2N}{\sqrt{\overline{v^2}}} \left[\ln\left(\frac{D\overline{v^2}}{3GM}\right) - 0.355\right] t \tag{56}$$

1.8 Relaxation time of a stellar system

The relaxation time t_R of a stellar system is the timescale over which dynamical relaxation occurs, or roughly the time over which accumulated perturbations to a star's velocity is on the same order as its velocity (essentially the system's dynamical "thermal time"). Adopting the standard definition that t_R is when $\overline{v^2} = \overline{\Delta v^2}$, then t_R is given by

$$t_R = \frac{1}{6}\left(\frac{3}{2\pi}\right)^{3/2} \frac{\left(\overline{v^2}\right)^{3/2}}{G^2M^2N\left[\ln\left(\frac{D\overline{v^2}}{3GM}\right) - 0.355\right]} \tag{57}$$

We can compare this expression for t_R to that which can be obtained by modeling the velocity perturbations using two-body encounters [10]:

$$t_R = \frac{1}{16}\left(\frac{3}{\pi}\right)^{1/2} \frac{\left(\overline{v^2}\right)^{3/2}}{G^2M^2N\ln\left(\frac{D\overline{v^2}}{3GM}\right)} \tag{58}$$

It is seen that the statistical framework presented by Chandrasekhar in [1] can recover a very similar result.

2 *Dynamical friction. I. General considerations: the coefficient of dynamical friction*

This Section follows Chandrasekhar's "Dynamical friction. I. General considerations: the coefficient of dynamical friction" [2]. This is the landmark work in which the concept of *dynamical friction* is first introduced, and treated in mathematical detail. Many of the mathematical details can also be found in Equation [11].

2.1 Issues with a pure random walk

Let $W_v(\vec{v},t)\,\mathrm{d}^3\vec{v}$ ($\mathrm{d}^3\vec{v} = \mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$) be the probability of finding v_x between v_x and $v_x + \mathrm{d}v_x$, v_y between v_y and $v_y + \mathrm{d}v_y$, and v_z between v_z and $v_z + \mathrm{d}v_z$. If the velocity obeys a random walk, then $W_v(\vec{v},t)$ obeys a diffusion equation of the form

$$\frac{\partial W_v}{\partial t} = q\nabla_{\vec{v}} W_v \tag{59}$$

where $\nabla_{\vec{v}}$ denotes a gradient in velocity space. We see that, if a star originally has a velocity $\vec{v}(t=0) = \vec{v}_0$ (i.e., $W(\vec{v},0;\vec{v}_0) = \delta^{(3)}(\vec{v}-\vec{v}_0)$), then $W_v(\vec{v},t;\vec{v}_0)$ will evolve as

$$W_v(\vec{v},t;\vec{v}_0) = \frac{1}{(4\pi qt)^{3/2}} \exp\left(-\frac{|\vec{v}-\vec{v}_0|^2}{4qt}\right) \tag{60}$$

Comparing Equation 60 above to Equation 41, we see that

$$q = \frac{1}{6}\overline{F^2 T} \tag{61}$$

However, we see that there are some difficulties at significantly long times t. At long times, we see that the distribution $W_v(\vec{v}, t)$ consistently broadens with time without bound. This brings the following conceptual issues:

- If the distribution of velocity fluctuations broadens without time to arbitrary width, then, given a long enough time, our model predicts that a star must experience arbitrarily high accelerations, which is unphysical.
- We expect that the distribution tend toward a Maxwellian distribution when $t \to \infty$. We see that the Maxwellian distribution is *not*, however, invariant under this framework.

We see then that our model only applies for times much shorter than a relaxation time, i.e., $t \ll t_R = \overline{v^2}/\overline{\Delta v^2} = \overline{v^2}/\overline{F^2 T}$ (using Equation 43). **The fact that accumulating stochastic accelerations may become arbitrarily large suggests the need for a damping force which moderates them at high velocities.**

2.2 Inferring dynamical friction from the invariance of the Maxwellian distribution

Consider a time interval $T(F) \ll \Delta t \ll t_R$, in other words, one during which the velocity does not change substantially but where the force on our star has been allowed to fluctuate many times. Then, from considerations from the study of Brownian motion (see [12] and, for application to stellar dynamics, [11]), the change in velocity $\Delta\vec{v}$ can be decomposed as

$$\Delta\vec{v} = \delta\vec{v}(\Delta t) - \eta\vec{v}\Delta t \tag{62}$$

where η is the *coefficient of dynamical friction*. Here, the first term represents a fluctuating part, whereas the latter term represents a systematic dynamical friction effect. While we note that we have written the effect of dynamical friction to be directly proportional to $\vec{v}$, $\eta = \eta(\vec{v})$ may, in general, depend on $\vec{v}$.

Then the probability distribution governing the stochastic part $\delta\vec{v}(\Delta t)$ is given by

$$\Psi(\delta\vec{v}[\Delta t]) = \frac{1}{(4\pi q\Delta t)^{3/2}} \exp\left(-\frac{|\delta\vec{v} - \nabla_{\vec{v}} q\Delta t|^2}{4qt}\right) \tag{63}$$

which is a version of Equation 60 which has been generalized to allow for $q = q(\vec{v})$.

We next follow [11] in deriving the Fokker-Planck equation for $W_v(\vec{v}, t)$. In particular, we an equation relating the velocity distribution at a later time $W_v(\vec{v}, t + \Delta t)$ with respect to its value at previous times, by adding up the probabilities of all the possible ways it could achieve a velocity $\vec{v}$ at a time $t + \Delta t$:

$$W_v(\vec{v}, t + \Delta t) = \int_{-\infty}^{+\infty} W_v(\vec{v} - \Delta\vec{v}, t)\Psi(\vec{v} - \Delta\vec{v}; \Delta\vec{v})\, \mathrm{d}^3\,(\Delta\vec{v}) \tag{64}$$

We Taylor expand $W_v(\vec{v}, t+\Delta t)$, $W_v(\vec{v}-\Delta\vec{v}, t)$, and $\Psi(\vec{v}-\Delta\vec{v};\Delta\vec{v})$ to first-order in Δt and second-order in $\Delta\vec{v}$. In particular:

$$
\begin{aligned}
W_v(\vec{v}, t+\Delta t) &= W_v(\vec{v}, t) + \frac{\partial W_v}{\partial t}\Delta t + \mathcal{O}\left(\Delta t^2\right) \\
W_v(\vec{v}-\Delta\vec{v}, t) &= W_v(\vec{v}, t) + \sum_{i=1}^{3}\frac{\partial W_v}{\partial v_i}\Delta v_i + \frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2 W_v}{\partial v_i^2}\Delta v_i^2 + \sum_{i=1}^{3}\sum_{j<i}\frac{\partial^2 W_v}{\partial v_i \partial v_j}\Delta v_i \Delta v_j + \mathcal{O}\left(\Delta v^3\right) \\
\Psi(\vec{v}-\Delta\vec{v};\Delta\vec{v}) &= \Psi(\vec{v};\Delta\vec{v}) + \sum_{i=1}^{3}\frac{\partial \Psi}{\partial v_i}\Delta v_i + \frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2 \Psi}{\partial v_i^2}\Delta v_i^2 + \sum_{i=1}^{3}\sum_{j<i}\frac{\partial^2 \Psi}{\partial v_i \partial v_j}\Delta v_i \Delta v_j + \mathcal{O}\left(\Delta v^3\right)
\end{aligned}
\tag{65}
$$

Letting

$$
\begin{aligned}
\overline{\Delta v_i} &= \int_{-\infty}^{+\infty}\Delta v_i \Psi(\vec{v};\Delta\vec{v})\,\mathrm{d}^3\left(\Delta\vec{v}\right) \\
\overline{\Delta v_i^2} &= \int_{-\infty}^{+\infty}\Delta v_i^2 \Psi(\vec{v};\Delta\vec{v})\,\mathrm{d}^3\left(\Delta\vec{v}\right) \\
\overline{\Delta v_i \Delta v_j} &= \int_{-\infty}^{+\infty}\Delta v_i \Delta v_j \Psi(\vec{v};\Delta\vec{v})\,\mathrm{d}^3\left(\Delta\vec{v}\right)
\end{aligned}
\tag{66}
$$

we can substitute Equations 65 (the Taylor expansions) into Equation 64 using Equations 66 to find:

$$
\begin{aligned}
\frac{\partial W_v}{\partial t} + \mathcal{O}\left(\Delta t^2\right) = &-\sum_{i=1}^{3} W_v \frac{\partial \overline{\Delta v_i}}{\partial v_i} - \sum_{i=1}^{3}\overline{\Delta v_i}\frac{\partial W_v}{\partial v_i} \\
&+ \frac{1}{2}\sum_{i=1}^{3} W_v \frac{\partial^2 \overline{\Delta v_i^2}}{\partial v_i^2} + \sum_{i=1}^{3}\frac{\partial W_v}{\partial v_i}\frac{\partial \overline{\Delta v_i^2}}{\partial v_i} + \frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2 W_v}{\partial v_i^2}\overline{\Delta v_i^2} \\
&+ \sum_{i=1}^{3}\sum_{j<i} W_v \frac{\partial^2 \overline{\Delta v_i \Delta v_j}}{\partial v_i \partial v_j} + \sum_{i=1}^{3}\sum_{j<i}\frac{\partial W_v}{\partial v_i}\frac{\partial \overline{\Delta v_i \Delta v_j}}{\partial v_j} + \sum_{i=1}^{3}\sum_{j<i}\frac{\partial^2 W_v}{\partial v_i \partial v_j}\overline{\Delta v_i \Delta v_j} + \mathcal{O}\left(\Delta v^3\right)
\end{aligned}
\tag{67}
$$

We note that the zeroth order terms in the expansion cancel out of both sides, as expected:

$$
W_v(\vec{v}, t) = W_v(\vec{v}, t)\int_{-\infty}^{+\infty}\Psi(\vec{v};\Delta\vec{v})\,\mathrm{d}^3\left(\Delta\vec{v}\right) \tag{68}
$$

by the normalization of $\Psi(\vec{v};\Delta\vec{v})$.

We notice that the other terms pair up in a product rule sense, and rewrite it as

$$
\begin{aligned}
\frac{\partial W_v}{\partial t}\Delta t + \mathcal{O}\left(\Delta t^2\right) = &-\sum_{i=1}^{3}\frac{\partial}{\partial v_i}\left(W_v \overline{\Delta v_i}\right) + \frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2}{\partial v_i^2}\left(W_v \overline{\Delta v_i^2}\right) \\
&+ \sum_{i=1}^{3}\sum_{j<i}\frac{\partial^2}{\partial v_i \partial v_j}\left(W_v \overline{\Delta v_i \Delta v_j}\right) + \mathcal{O}\left(\Delta v^3\right)
\end{aligned}
\tag{69}
$$

Equation 69 above is considered the *general Fokker-Planck equation*, but can be made specific using our explicit expression for $\Psi(\vec{v};\Delta\vec{v})$ (Equation 63). In particular, we have

$$\begin{aligned}\overline{\Delta v_i} &= -\left(\eta v_i - \frac{\partial q}{\partial v_i}\right)\Delta t \\ \overline{\Delta v_i^2} &= 2q\Delta t \\ \overline{\Delta v_i \Delta v_j} &= \mathcal{O}\left(\Delta t^2\right) \text{ when } i \neq j\end{aligned} \tag{70}$$

with moments higher order in Δv being at least $\mathcal{O}\left(\Delta t^3\right)$. The Fokker-Planck equation then becomes

$$\frac{\partial W_v}{\partial t} + \mathcal{O}\left(\Delta t^2\right) = \sum_{i=1}^{3} \frac{\partial}{\partial v_i}\left(\eta v_i W_v - W_v \frac{\partial q}{\partial v_i}\right) + \sum_{i=1}^{3} \frac{\partial^2}{\partial v_i^2}\left(q W_v\right) + \mathcal{O}\left(\Delta t^2\right) \tag{71}$$

Then, equating first-order terms, we obtain a kind of continuity equation for $W_v(\vec{v},t)$ in velocity space:

$$\frac{\partial W_v}{\partial t} = \nabla_{\vec{v}} \cdot \left(\eta \vec{v} W_v + q \nabla_{\vec{v}} W_v\right) \tag{72}$$

With this equation in hand, we next write down a Maxwellian distribution at $t = 0$:

$$W_v(\vec{v},0) = \left(\frac{3}{2\pi\overline{v^2}}\right)^{3/2} \exp\left(-\frac{3v^2}{2\overline{v^2}}\right), \tag{73}$$

where we enforce that $W_v(\vec{v},t) = W_v(\vec{v},0)$ for the Maxwellian specifically (i.e., a Maxwellian distribution of velocities will not vary with time, $\partial W_v/\partial t = 0$). This can be enforced if

$$\eta \vec{v} W_v = -q \nabla_{\vec{v}} W_v \tag{74}$$

Since $\nabla_{\vec{v}} W_v = -3\vec{v} W_v/\overline{v^2}$, this implies that η must be related to q as

$$\eta = \frac{3}{\overline{v^2}} q \tag{75}$$

where $\overline{v^2}$ is a constant.

Now, if again the initial velocity of our star is $\vec{v}_0$ (i.e., $W_v(\vec{v},0) = \delta^{(3)}(\vec{v} - \vec{v}_0)$), then we can solve the Fokker-Planck equation formally as

$$W_v(\vec{v},t;\vec{v}_0) = \left[\frac{3}{2\pi\overline{v^2}\left(1 - e^{-2\eta t}\right)}\right]^{3/2} \exp\left[-\frac{3\left|\vec{v} - \vec{v}_0 e^{-\eta t}\right|^2}{2\overline{v^2}\left(1 - e^{-2\eta t}\right)}\right] \tag{76}$$

where we have now assumed q and η are constants with $\vec{v}$. We notice that Equation 76 above becomes Equation 60 when $t \ll \eta^{-1}$, i.e., when not enough time has elapsed for the effect of dynamical friction to have become noticeable. Moreover, as it tends towards a Maxwellian distribution at $t \to \infty$, this solution resolves our aforementioned issue with the $\eta = 0$ solution. Since η^{-1} thus defines a timescale over which the diffusive behavior of $\vec{v}$ gives over to a Maxwellian distribution, it can be considered a metric of the system's *relaxation time*. This can be seen explicitly from Equations 75 and 43:

$$\eta^{-1} = \frac{2\overline{v^2}}{\overline{F^2}T} = 2t_R \tag{77}$$

2.3 Deriving η from two-body encounters

We can calculate η by considering a series of two-body encounters meant to model fluctuations in $\vec{F}$ felt by our star, which we will now take to have mass M_2 and velocity $\vec{v}_2$. Consider a two-body encounter with another star from the field of mass M_1 and velocity $\vec{v}_1$. The changes $\Delta v_\parallel$ and $\Delta v_\perp$ it will experience to its velocity in the directions parallel and perpendicular (respectively) to its direction of motion are given in Chandrasekhar's *Principles of Stellar Dynamics* [10] as

$$\begin{aligned}\Delta v_\parallel &= -\frac{2M_1}{M_1+M_2}\left[(v_2 - v_1\cos\theta)\cos\psi + v_1\sin\theta\cos\Theta\sin\psi\right]\cos\psi \\ \Delta v_\perp &= \pm\frac{2M_1}{M_1+M_2}\left[v_1^2+v_2^2-2v_1v_2\cos\theta - \left[(v_2-v_1\cos\theta)\cos\psi + v_1\sin\theta\cos\Theta\sin\psi\right]^2\right]^{1/2}\cos\psi\end{aligned} \tag{78}$$

where [10] (page 51) defines the angles in the above expressions as follows:

- θ is the angle between $\vec{v}_1$ and $\vec{v}_2$
- Θ is the angle between the orbital plane and fundamental plane which contains both the vectors $\vec{v}_1$ and $\vec{v}_2$
- $\pi - 2\psi$ is the angle of deflection of the relative velocities of the two stars, and is given by

$$\cos\psi = \frac{1}{\sqrt{1+\frac{D^2V^4}{G^2(M_1+M_2)^2}}} \tag{79}$$

  for an impact parameter D and relative velocity V.

Over many encounters, $\Delta v_\perp$ vanishes by symmetry (since there is no preferential perpendicular direction for the star to be scattered). However, $\Delta v_\parallel$ will accumulate in general. Consider a time interval Δt as before satisfying $T(F) \ll \Delta t \ll t_R$. Then we may find the total perturbation to the parallel component of our star's velocity by integrating over possible orbital parameters:

$$\sum_i \Delta v_{\parallel,i} = \Delta t \int_0^\infty \mathrm{d}v_1 \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \int_0^{D_0} \mathrm{d}D \int_0^{2\pi} \frac{\mathrm{d}\Theta}{2\pi}\left(2\pi N(v_1,\theta,\varphi) V D \Delta v_\parallel\right) \tag{80}$$

where φ is the azimutal angle in a spherical coordinate system where $\vec{v}_2$ points along the $+z$ direction, D_0 is a maximum impact parameter, and $N(v_1,\theta,\varphi)$ is a weight function for the relative occurrence of scattering events with parameters v_1, θ, and φ.

Integrating over Θ, we have

$$\sum_i \Delta v_{\parallel,i} = -4\pi\frac{M_1}{M_1+M_2}\Delta t \int_0^\infty \mathrm{d}v_1 \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \int_0^{D_0} \mathrm{d}D\, N(v_1,\theta,\varphi)\frac{V(v_2 - v_1\cos\theta)D}{1 + D^2V^4/G^2(M_1+M_2)^2} \tag{81}$$

Chandrasekhar comments in [10] on the choice of D_0 as being within a factor of a few of the typical spacing between stars, in order to prevent in the D integral in Equation 81 from diverging. The basic argument describes an overestimate of the exchanged energy at large impact parameters, where the stars begin and end an infinite distance away from each other and a two-body encounter approach is inappropriate.

Integrating over D, we now have

$$\sum_i \Delta v_{\|,i} = -2\pi M_1(M_1+M_2)G^2\Delta t \int_0^\infty \mathrm{d}v_1 \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi\, N(v_1,\theta,\varphi) \frac{(v_2 - v_1\cos\theta)\ln\left(1+\mathcal{V}^2V^4\right)}{V^3} \tag{82}$$

where

$$\mathcal{V} = \frac{D_0}{G(M_1+M_2)} \tag{83}$$

Now, if the distribution of $\vec{v}_1$ is assumed to be spherically symmetric, then

$$N(v_1,\theta,\varphi) = N(v_1)\frac{1}{4\pi}\sin\theta \tag{84}$$

and

$$\Delta v_\| = -\pi M_1(M_1+M_2)G^2\Delta t \int_0^\infty \mathrm{d}v_1\, N(v_1) \int_0^\pi \mathrm{d}\theta\, \frac{\sin\theta}{V^3}(v_2 - v_1\cos\theta)\ln\left(1+\mathcal{V}^2V^4\right) \tag{85}$$

Note by the law of cosines that $V^2 = v_1^2 + v_2^2 - 2v_1v_2\cos\theta$, so that $V\,\mathrm{d}V = v_1v_2\sin\theta\,\mathrm{d}\theta$, and also $v_2 - v_1\cos\theta = \left(V^2+v_2^2-v_1^2\right)/2v_2$. Then our integral becomes

$$\sum_i \Delta v_{\|,i} = -\frac{1}{2}\pi M_1(M_1+M_2)\frac{G^2}{v_2^2}\Delta t \int_0^\infty \frac{1}{v_1}N(v_1)\mathcal{J}\,\mathrm{d}v_1 \tag{86}$$

where $\mathcal{J}$ denotes

$$\mathcal{J} = \int_{|v_1-v_2|}^{(v_1+v_2)} \left(1+\frac{v_2^2-v_1^2}{V^2}\right)\ln\left(1+\mathcal{V}^2V^4\right)\,\mathrm{d}V \tag{87}$$

Applying an integration by parts yields

$$\mathcal{J} = \left(V - \frac{v_2^2-v_1^2}{V}\right)\ln\left(1+\mathcal{V}^2V^4\right)\Big|_{|v_1-v_2|}^{(v_1+v_2)} \left(1-\frac{v_2^2-v_1^2}{V^2}\right)\frac{\mathcal{V}^2V^4}{1+\mathcal{V}^2V^4}\,\mathrm{d}V \tag{88}$$

Note that usually $\mathcal{V}^2V^4 \gg 1$ for physical situations. This is equivalent to taking $V \gg \sqrt{G(M_1+M_2)/D_0}$; it seems clear that, in realistic stellar systems, the typical stellar velocity is much greater than the escape velocity from the system at a distance comparable to the stellar separations (i.e., uniformly distributed stars should not generally be expected to be bound to each other).

Then Equation 88 is very nearly

$$\mathcal{J} = \left(V - \frac{v_2^2-v_1^2}{V}\right)\ln\left(1+\mathcal{V}2V^4\right) - 4\left(V+\frac{v_2^2-v_1^2}{V}\right)\Big|_{|v_1-v_2|}^{(v_1+v_2)} \tag{89}$$

This expression can be evaluated on a case-by-case basis to find

$$\mathcal{J} = \begin{cases} 2v_1 \ln\left[\left(1+\mathcal{V}^2(v_1+v_2)^4\right)\left(1+\mathcal{V}^2(v_1-v_2)^4\right)\right] & v_1 < v_2 \\ 2v_1 \ln\left(1+16\mathcal{V}^2 v_1^4\right) - 8v_1 & v_1 = v_2 \\ 2v_1 \ln\left(\frac{1+\mathcal{V}^2(v_1+v_2)^4}{1+\mathcal{V}^2(v_1-v_2)^4} - 16v_2\right) & v_1 > v_2 \end{cases} \tag{90}$$

Again noting that $\mathcal{V}^2(v_1+v_2)^4, \mathcal{V}^2(v_1-v_2)^4 \gg 1$, we have

$$\mathcal{J} = \begin{cases} 8v_1 \ln\left(\mathcal{V}\left(v_2^2 - v_1^2\right)\right) & v_1 < v_2 \\ 4v_1 \ln\left(4\mathcal{V}v_1^2\right) - 8v_1 & v_1 = v_2 \\ 8v_1 \ln\left(\frac{v_1+v_2}{v_1-v_2}\right) - 16v_2 & v_1 > v_2 \end{cases} \tag{91}$$

We can very roughly retain the dominant term, whereby we obtain

$$\mathcal{J} = \begin{cases} 8v_1 \ln\left(\mathcal{V}\overline{v^2}\right) & v_1 < v_2 \\ 0 & v_1 > v_2 \end{cases} \tag{92}$$

where, for large $\mathcal{V}$, $\mathcal{J}$ is larger in the $v_1 < v_2$ case is larger than the $v_1 < v_2$ case, and $v_2^2 - v_1^2 \simeq \overline{v^2}$ for fixed v_2. Through this approximation, we see that encounters with $v_1 < v_2$ (encounters slower than or comparable to the speed of our star) contribute most strongly to the dynamical friction. This observation is crucial for identifying the cause of dynamical friction.

Adopting this approximation, we now have

$$\sum_i \Delta v_{\parallel,i} = -4\pi M_1(M_1+M_2)\frac{G^2}{v_2^2}\ln\left(\mathcal{V}\overline{v^2}\right)\Delta t \int_0^{v_2} N(v_1)\,\mathrm{d}v_1 \tag{93}$$

which contains the integral

$$\int_0^{v_2} N(v_1)\,\mathrm{d}v_1 = \frac{4j^3}{\sqrt{\pi}} N \int_0^{v_2} e^{-j^2 v_1^2} v_1^2\,\mathrm{d}v_1 \tag{94}$$

if $N(v_1)$ is assumed to be Maxwellian (the distribution of speeds derivable from Equation 73). Then we have

$$\sum_i \Delta v_{\parallel,i} = -4\pi N M_1(M_1+M_2)\frac{G^2}{v_2^2}\ln\left(\mathcal{V}\overline{v^2}\right)\Delta t \left[\operatorname{erf}(jv_2) - \frac{2jv_2}{\sqrt{\pi}}e^{-jv_2}\right] \tag{95}$$

Remembering that $\eta v_2 = -\sum_i \Delta v_{\parallel,i}$, we have

$$\eta = 4\pi N M_1(M_1+M_2)\frac{G^2}{v_2^3}\ln\left(\mathcal{V}\overline{v^2}\right)\Delta t \left[\operatorname{erf}(jv_2) - \frac{2jv_2}{\sqrt{\pi}}e^{-jv_2}\right] \tag{96}$$

Note that the quantity $\ln\left(\mathcal{V}\overline{v^2}\right) \equiv \ln\Lambda$ is frequently referred to as the "Coulomb logarithm," and can be interpreted as the logarithm of the ratio between the largest and smallest possible impact parameters (D_0 and $2GM/v^2$, respectively).

A lengthy calculation in *Principles of Stellar Dynamics* [10] gives the sum of squared velocity perturbations:

$$\sum_i \Delta v_{\parallel,i}^2 = \frac{8}{3}\pi N M_1^2 \frac{G^2}{v_2^3}\overline{v^2}\ln\left(\mathcal{V}\overline{v^2}\right)\Delta t\left[\operatorname{erf}(jv_2) - \frac{2jv_2}{\sqrt{\pi}}e^{-jv_2}\right] \tag{97}$$

We find that

$$\frac{\sum_i \Delta v_{\parallel,i}^2}{\eta\Delta t} = \frac{2}{3}\frac{M_1}{M_1+M_2}\overline{v^2} \tag{98}$$

which reproduces the expected $q/\eta = \overline{v^2}/3$ (Equation 75) in the equal mass case.

2.4 The non-spherical case

The preceding analysis of dynamical friction made use of spherical symmetry, which may not be generically true. In particular, we can generalize the distribution $\Psi(\delta\vec{v}[t])$ (Equation 63) to

$$\Psi(\delta\vec{v}[\Delta t]) = \frac{|a|}{\pi^{3/2}}\exp\left(-\frac{1}{\Delta t}\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\delta v_i \delta v_j\right) \tag{99}$$

where a_{ij} are the components of a symmetric matrix with determinant $|a|$. At the time of publication of [2], some progress had been made on the direction of determining a_{ij} using a statistical approach analyzing fluctuations in $\dot{\vec{F}}$ by [13, 14], where an indication of dynamical friction was recovered.

In the last two pages of [2], Chandrasekhar outlines some basic considerations which underscore the development of a more general statistical theory. In particular, one seeks the joint distribution function $W(\vec{F}_0, \vec{F}_t)$ which relates how the force $\vec{F}_0$ at a given time $t = 0$ is related to the force $\vec{F}_t$ at a later time t, where we consider times short enough that the stars have roughly followed linear trajectories:

$$\vec{F}_t = G\sum_i M_i \frac{\vec{r}_i + \vec{v}_i t}{|\vec{r}_i + \vec{v}_i t|^3} \tag{100}$$

Formally, [13, 14] find that

$$W(\vec{F}_0, \vec{F}_t) = \frac{1}{64\pi^6}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left(-i\left(\vec{w}_1\cdot\vec{F}_0 + \vec{w}_2\cdot\vec{F}_t\right)\right)A\left(\vec{w}_1,\vec{w}_2\right)\mathrm{d}\vec{w}_1\mathrm{d}\vec{w}_2 \tag{101}$$

where

$$\begin{aligned} A\left(\vec{w}_1,\vec{w}_2\right) &= \exp\left(-NC\left(\vec{w}_1,\vec{w}_2\right)\right) \\ C\left(\vec{w}_1,\vec{w}_2\right) &= \int_{-\infty}^{+\infty}\mathrm{d}\vec{v}\int_{-\infty}^{+\infty}\mathrm{d}\vec{r}\int_0^{+\infty}\mathrm{d}\vec{v}\,M\left[1-\exp\left(iGM\left(\frac{\vec{r}\cdot\vec{w}_1}{|\vec{r}|^3} + \frac{(\vec{r}+\vec{v}t)}{|\vec{r}+\vec{v}t|^3}\cdot\vec{w}_2\right)\right)\right]\tau(\vec{v},M) \end{aligned} \tag{102}$$

Here, $\tau(\vec{v}, M)$ describes the probability of occurrence of a star of velocity $\vec{v}$ and mass M. Chandrasekhar claims that it is sufficient to know only the first moment of $\vec{F}(t)$ (i.e., $\overline{\vec{F}_t}$), which corresponds to knowing only $C(\vec{w}_1, \vec{w}_2)$ in the limit of large $\vec{w}_2$. Chandrasekhar notes that studying the integral

$$\int_0^\infty \overline{\vec{F}_t}(\vec{F}_0, \vec{v}) \, \mathrm{d}t \tag{103}$$

will reveal both a more precise meaning of the force fluctuation lifetime T as well as the role of dynamical friction within the statistical theory.

2.5 Differences from fluid viscosity

Chandrasekhar notes that concept of dynamical friction is comparable to, but distinct from that of fluid viscosity in the following senses:

- Whereas dynamical friction acts on individual particles (stars), viscosity describes an effect between gas *elements*.
- While dynamical friction describes an exact damping effect which acts on moving stars, viscosity is a phenomenon which is only sensibly defined on timescales longer than a relaxation time.

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