Rotating stars in general relativity

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For anyone who's further interested:

- Paschalidis and Stergioulas (2017) : A broad review on this topic.
- Chapter 11 & 12 of Rezzolla and Zanotti (2013) : section 11.6-7 on non-self gravitating accretion disks and section 12.3 on rotating compact stars.
- the author's office (Cahill 355) : doesn't help much, but I'll serve a nice coffee :)

I Newtonian problem

- How gravitational field is generated :

$$\nabla^2 \Phi = 4\pi G \rho \tag{1}$$

- How matter behaves on it :

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{2}$$

Assume stationary state + axial symmetry:

$$-\Omega^2 \vec{\varpi} = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{3}$$

which is no other than a force balance equation. Taking the curl ($\nabla \times$) on both sides of (3), we have

$$(\ldots)\, \vec{\varpi} \times \nabla \Omega = (\ldots)\, \nabla \rho \times \nabla p \tag{4}$$

Therefore, if the equation of state of a rotating star is barotropic $p = p(\rho)$, the rotation profile $\Omega(r, \theta)$ of it must only depend on the cylindrical radius i.e. $\Omega(r, \theta) = \Omega(r \sin\theta) = \Omega(\varpi)$. This is known as the Poincare conditions or von Zeipel's theorem (Zeipel, 1924).

II General relativity

II.1 EQUATIONS

- Matter tells space how to curve :

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{5}$$

- and space tells matter how to move :

$$\nabla_\mu T^{\mu\nu} = 0 \tag{6}$$

For stationary and axisymmetric spacetime without any meridional motions of the fluid, we can write out the metric as

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \\ &= -e^{\gamma+\rho}dt^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2) + e^{\gamma-\rho}r^2 \sin^2\theta(d\phi - \omega dt)^2 \end{aligned} \tag{7}$$

where the metric functions $\gamma, \rho, \alpha, \omega$ depend only on r and θ coordinates.
The 4-velocity of fluid can be written as

$$u^\mu = u^t(1, 0, 0, \Omega) \tag{8}$$

where $\Omega \equiv u^\phi/u^t$ is coordinate angular velocity. Normalization $u^\mu u_\mu = -1$ gives

$$-\left(u^t\right)^{-2} = g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 \tag{9}$$

so we can get

$$u^\mu = \frac{e^{-(\rho+\gamma)/2}}{\sqrt{1-V^2}}(1, 0, 0, \Omega) \tag{10}$$

with $V = (\Omega - \omega)e^{-\rho}r\sin\theta$. When the non-isotropic stresses, viscosity, and heat transport can be neglected, we can model the matter as a perfect fluid

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{11}$$

Explicitly writing down gravitational field equations is tedious but straightforward; here we omit the details but just make two small remarks. First, it is easier to use the Einstein equation in an equivalent form

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{12}$$

and it seems that many literatures have actually made this choice for formulating field equations. Second, these equations can be written as three elliptic equations

$$\begin{aligned} \nabla^2[\rho e^{\gamma/2}] &= S_\rho(g, \partial_\mu g, T) \\ \nabla^2[\gamma e^{\gamma/2}] &= S_\gamma(g, \partial_\mu g, T) \\ \nabla^2[\omega e^{(\gamma-2\rho)/2}] &= S_\omega(g, \partial_\mu g, T) \end{aligned} \tag{13}$$

with another first-order PDE

$$\frac{\partial\alpha}{\partial\theta} = S_\alpha(g, \partial_\mu g, \partial_\mu^2 g) \tag{14}$$

for the metric functions.
Plugging (11) into (6) we have

$$-\partial_\mu \ln|u^t| + u^t u_\phi \partial_\mu\Omega = -\frac{\partial_\mu p}{\epsilon + p} \tag{15}$$

which is sometimes called as relativistic Euler equations. If $u^t u_\phi = j(\Omega)$, this equation is integrable and $p = p(\epsilon)$. Differently speaking, for a rotating fluid body following barotropic equation of states, $u^t u_\phi$ is a function of Ω. It can be also shown that this is equivalent to the statement that $l \equiv -u_\phi/u_t$ is a function of Ω, which is a general relativistic version of the von Zeipel's theorem (Abramowicz, 1971).

II.2 NEUTRON STAR MODELS

For neutron stars we may use the equation of states of cold matter, which is one-paramter and $p = p(\epsilon)$; this is a valid assumption since the typical Fermi energy of the neutron degeneracy is the order of MeV $\sim 10^{10}$K for neutron stars.

A nascent neutron star is expected to be differentially rotating, but several physical mechanisms drives angular velocity distribution to be uniform i.e. $\Omega =$ const. An estimate from kinematical shear viscosity is (Cutler and Lindblom, 1987) :

$$\tau \sim 18 \left(\frac{\rho}{10^{15}\mathrm{g\,cm^{-3}}}\right)^{-5/4} \left(\frac{T}{10^{9}\mathrm{K}}\right)^{2} \left(\frac{R}{10^{6}\mathrm{cm}}\right)^{2} \mathrm{yr} \tag{16}$$

In the presence of magnetic fields, magnetic braking or magneto-rotational instability (MRI) may decrease this timescale to the order of 1 second.

Therefore, an isolated rotating neutron star can be modeled fairly accurately by an uniformly rotating fluid which follows barotropic equation of states.

III How to solve them?

For non-rotating, spherically symmetric compact stars (TOV stars) there exist a few analytic solutions, though they usually give up the physical validity of the equation of states (e.g. infinite sound speed). One might take a look at this arxiv article for some examples. Still, the TOV equations are fairly simple and can be solved by using common ODE integrators. Birkhoff's theorem guarantees that the spacetime geometry outside a spherically symmetric star is always the Schwarzschild solution with an appropriate value of M.

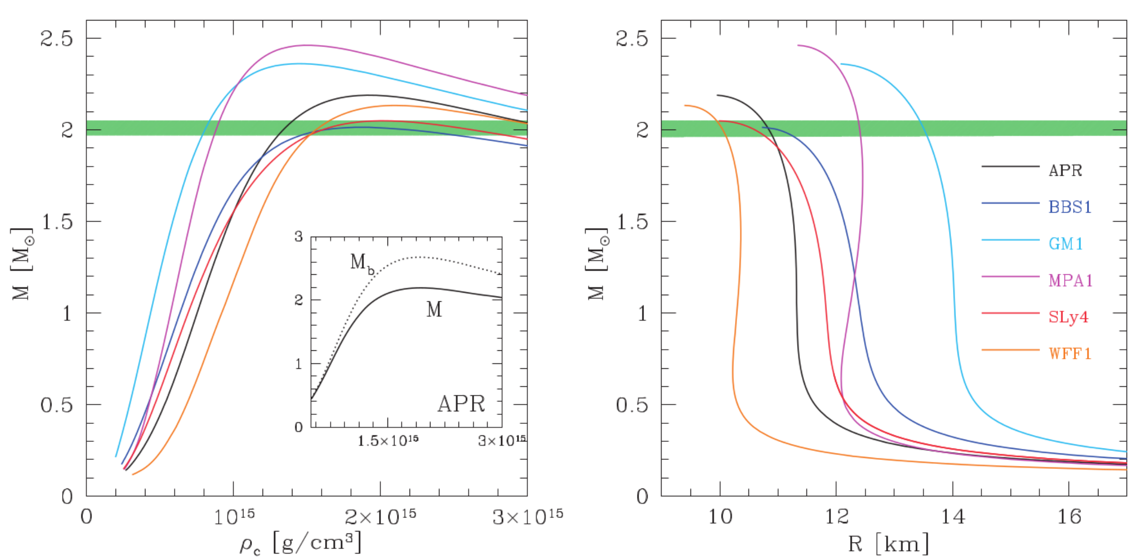

Figure 1: from Rezzolla and Zanotti (2013). Numerical solutions of TOV equations for a number of EOSs

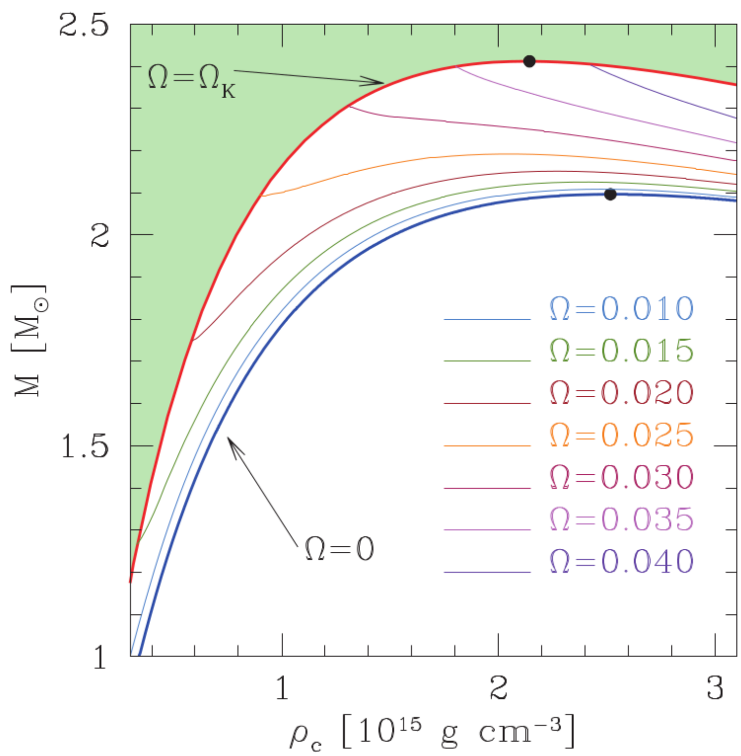

Figure 2: also from Rezzolla and Zanotti (2013). Equilibrium solutions of uniformly rotating neutron stars.

For rotating compact stars no closed-form analytic solution has been found yet and the equations need to be solved numerically. Also the exterior geometry of a rotating fluid body is not the Kerr metric, which needs to be solved together at the same time.

Some examples of numerical schemes are (sorry I didn't want to find all bib refs)

- Butterworth and Ipser (1976)
- Komatsu-Eriguchi-Hachisu (1989); Cook-Shapiro-Teukolsky (1992)
- Bonazzola-Gourgoulhon-Salgado-Marck (1993); the LORENE library (Gourgoulhon et al.)
- Ansorg-Kleinwächter-Meinel (2002)
- the COCAL code (Uryu et al)

There have been also several studies for direct comparisons between each codes (e.g. Nozawa et al., 1998).

IV So what does rotation do?

IV.1 MASS-SHEDDING LIMIT

The star would shed mass if it is spinning too fast; this angular frequency is called mass-shedding (or breakup, or Keplerian) frequency Ω_K. At this frequency the motion of the fluid element on the surface of the star on the equator is geodesic ($\partial_\mu p = 0$). Mass-shedding frequency is obviously a model-dependent quantity, and the fastest spinning pulsar gives an observational constraint on this. The highest rate observed so far is 716 Hz (Hessels et al., 2006).

IV.2 ON THE MAXIMUM MASS OF NS

The TOV mass limit $M_{\max}(\Omega = 0)$ is the most commonly referred one for the term "neutron star mass limit". However, for any value of the central density ρ_c the mass of the neutron star model increases up to $\sim$15% (yet depends on models) as the rotating angular frequency Ω increases, since the centrifugal force gives extra support against gravity. Therefore one needs to pay attention when interpreting or using the term "mass limit" whether it means the TOV mass limit or the *absolute* mass limit which also took rapidly rotating cases into consideration.

Rotating compact star models with $M > M_{\max}(\Omega = 0)$ are called *supramassive* stars. A supramassive star might spin down, reach instability, and collapse to a black hole.

If we consider differential rotation, the mass limit can further increase. A star with a mass larger than the maximum mass of uniformly rotating star is called a *hypermassive* star. A hypermassive star would undergo dissipative processes redistributing angular momentum and is forced collapse to a black hole. A metastable hypermassive neutron star (HMNS) is thought to be formed in binary neutron star mergers.

References

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