Ingredients for a BBH Simulation

Kyle Nelli

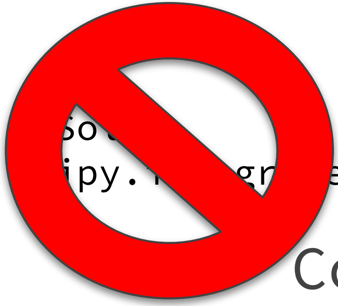
Ingredients

- Formulation of Einstein’s Equations
  - Determines most everything below
- Gauge Choice
- Initial Data
- Boundary Conditions
- Numerical Implementation
- Waveform Extraction

Formulation of Einstein's Equations

What is a BBH simulation at its core?

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$



`so`
`ipy. gr e.solve_ivp`

Coupled, second order, non-linear,

partial differential equations

How do we want the equations to look?

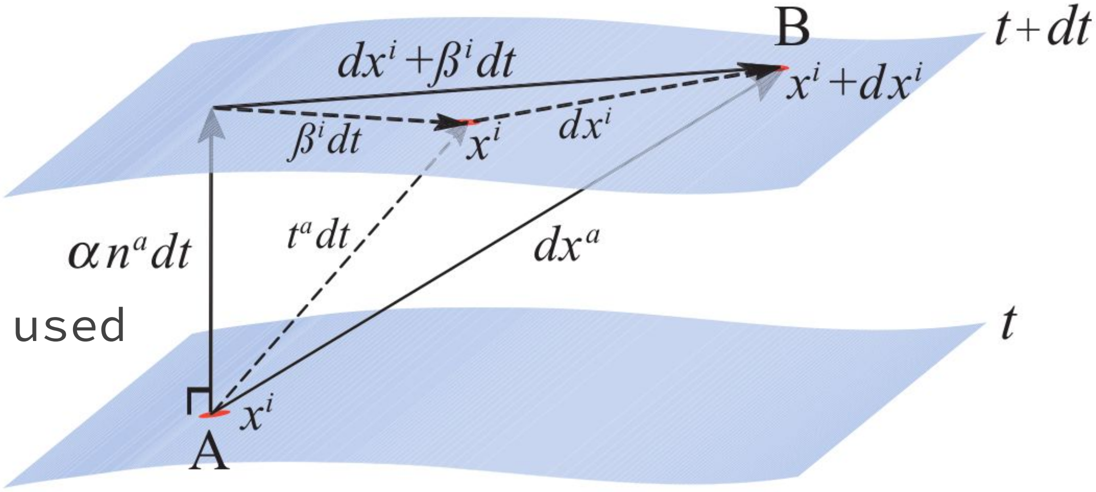
$$\partial_t \vec{u} + \mathbf{A}\partial_i \vec{u} = D(\vec{u})$$

- 1st order!
- $\vec{u}$ is a vector of things we are evolving
- $\mathbf{A}$ is a matrix
- $D(\vec{u})$ is source function
- Plus some constraint equations, like from E&M $\nabla \cdot \vec{B} = 0$

Formulations throughout history

- ADM (Arnowitt, Deser, Misner)
- BSSN (Baumgarte, Shapiro, Shibata, Nakamura)
- GH (Generalized Harmonic)
- Z4 family

ADM (early 1960s)



- ➔ First major formalism used
- ➔ 3+1 Decomposition
  - ◆ Spatial metric
  - ◆ Extrinsic curvature
  - ◆ Lapse, Shift
  - ◆ Derivatives
- ➔ Weakly hyperbolic -> Not well posed -> Unstable!!!

BSSN

- Similar to ADM
  - 3+1 Decomposition
- But add 2 new things
  - Conformal decomposition
  - New evolution variable

$$ds^2 = \psi^2 d\bar{s}^2$$

- Hyperbolicity now depends on gauge choice!
- Most widely used formulation (Einstein toolkit)

GH

- ➔ Does *NOT* use 3+1 decomposition
- ➔ Evolves entire spacetime (the metric itself)
- ➔ Strongly hyperbolic regardless of gauge
- ➔ Has a known characteristic decomposition
  - Eigenvectors and values of $\mathbf{A}$

$$\partial_t \vec{u} + \mathbf{A}\partial_i \vec{u} = D(\vec{u})$$

Formulation

This is what SXS uses :)

Z4 Family

- Very similar to BSSN
  - 3+1 decomposition with conformal decomposition
- Extra conformal transformations on constraints
- Now strongly hyperbolic!
  - Like GH
- “Best of both worlds”
- Many different formulations
  - Z4
  - CCZ4
  - Z4c
  - FO-CCZ4

Gauge Choice

Your formulation determines your gauge (coordinates)

- GH
  - Harmonic coordinates

$$\nabla_b \nabla^b x^a = 0 \implies -g^{bc}\Gamma^a_{bc} \equiv \Gamma^a = 0$$

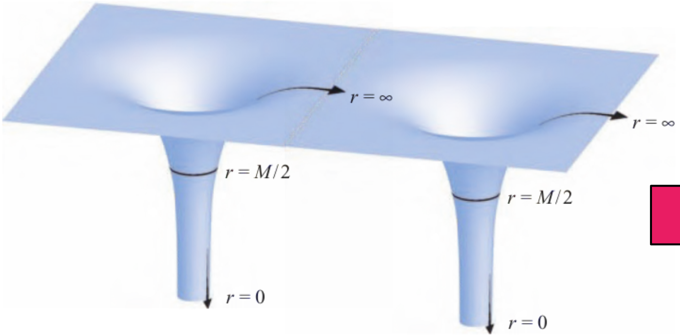
  - Generalized Harmonic coordinates

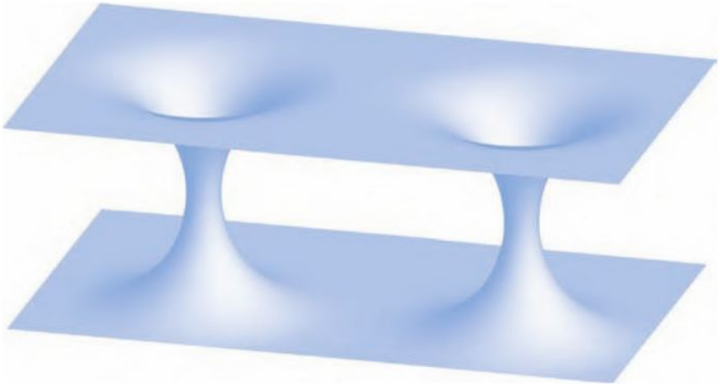
$$-\Gamma^a = H^a(t, x^i, g_{bc})$$

  - Still have singularities... :(
  - Probably have to use excision

Your formulation determines your gauge (coordinates)

- BSSN and Moving puncture gauge
  - Remove physical singularity from coordinates but leaves coordinate singularities
  - Just don’t put a grid point at the coordinate singularity...





Gauge Choice

Numerical Relativity: Solving Einstein’s Equations on the Computer, Baumgarte and Shapiro 2010

Initial Data

Need a snapshot of an evolution

- 1D wave equation, simple profile
- For BBH, need values and derivatives of metric everywhere
- Amounts to solving the constraint equations (elliptic equations)

$$R + K^2 + K_{ij}K^{ij} = 0$$

$$D_j(K^{ij} - \gamma^{ij}K) = 0$$

Initial Data

Methods

- ➔ Conformal transformations
  - Spatial metric
  - Extrinsic curvature
- ➔ Conformal transverse traceless decomposition (CTT)
  - Extrinsic curvature
- ➔ Conformal thin sandwich(CTS)
  - Gives time derivative of spatial metric also!
- ➔ Extended conformal thin sandwich (XCTS)
- ➔ Not unique to formulation of EEs
- ➔ Iterative

Superposition?... Superposition!! (well sort of...)

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

$$\bar{D}^2 \psi = \ldots$$

Boundary Conditions

Boundary conditions are hard ... :(

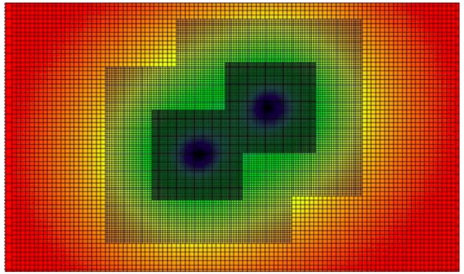
- ➔ Outflow
  - Outgoing characteristics
- ➔ Constraint preserving
  - Outgoing constraint violations
- ➔ Gauge
  - Outgoing gauge perturbations
- ➔ Physical
  - Ingoing characteristics
- ➔ Analytic
  - Asymptotic Flatness
- ➔ Big domain...

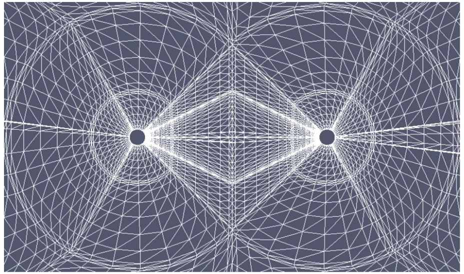
Numerical Implementation

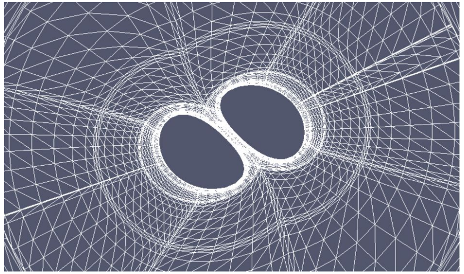
Technical things which are worth mentioning but aren't critical to your understanding

---

- ➔ Mesh
  - ◆ Cartesian, Curved, Moving
  - ◆ Mesh refinement
- ➔ Representation
  - ◆ Finite difference
  - ◆ Spectral
- ➔ Observations
  - ◆ Volume data
  - ◆ Constraints
  - ◆ Apparent horizons
  - ◆ Waveforms ***
- ➔ Time steppers/integrators







https://www.dam.brown.edu/ACCGR/ACCGR_talks/accgr_Frank_loeffler.pdf
https://arxiv.org/abs/1211.6079

Numerical Implementation

Waveform Extraction

Part 1: How to get usable quantities?

- ➔ Newman-Penrose
  - ◆ Weyl Scalars (from Weyl tensor)
  - ◆ Particularly,
- ➔ Moncrief formalism
  - ◆ Perturbative decomposition of metric
  - ◆ Even-odd parity parts

$$\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$$

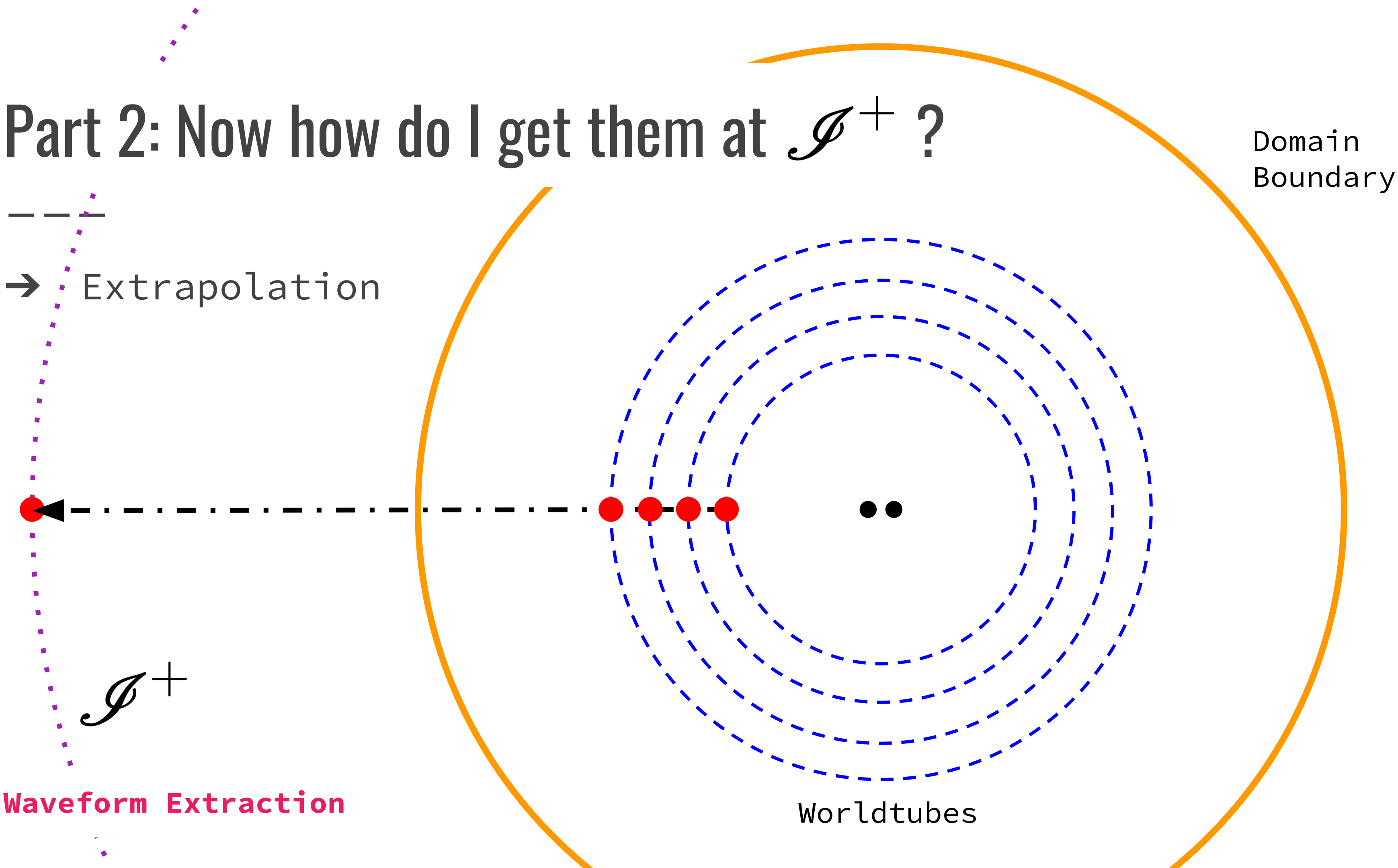
$$\psi_4 = \ddot{h}_+ - i\ddot{h}_\times$$

$$g_{ab} = g_{ab}^{\mathrm{Schw}} + h_{ab}$$

$$h_{ab} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left({}^{\mathrm{e}}h_{ab}^{\ell m} + {}^{\mathrm{o}}h_{ab}^{\ell m} \right)$$

Part 2: Now how do I get them at $\mathscr{I}^+$?

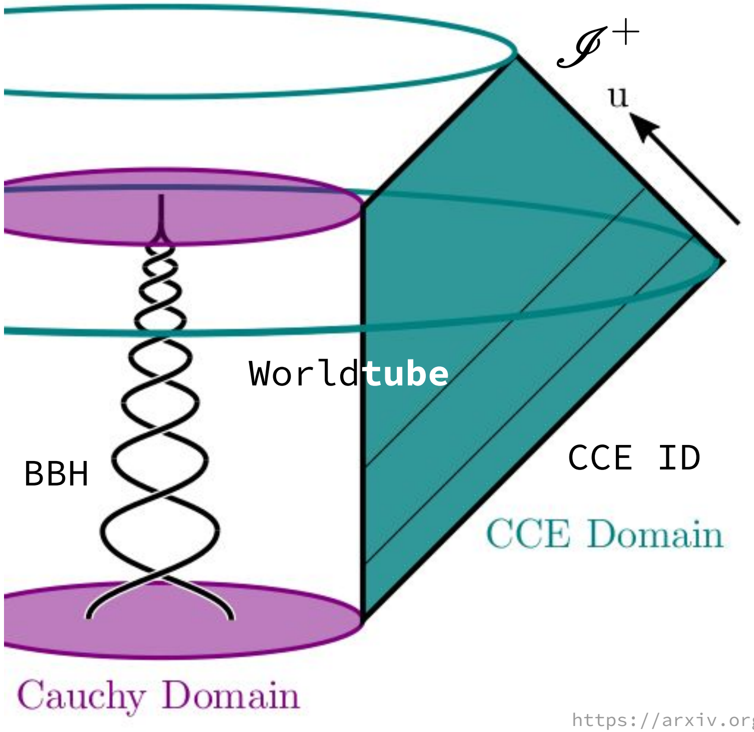
→ Extrapolation



Waveform Extraction

Part 2: Now how do I get them at $\mathscr{I}^+$?

➔ Cauchy Characteristic Extraction (CCE)



Waveform Extraction

https://arxiv.org/abs/2110.08635

Now you're ready to cook up some BBHs!

Ingredients

- → Formulation of Einstein's Equations
- → Gauge Choice
- → Initial Data
- → Boundary Conditions
- → Numerical Implementation
- → Waveform Extraction

Thank you!