California Institute of Technology
Physics 2a Recitation
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Practice Problem: "Wired and Loopy"
30 September 2022

"If you cut corners you just keep on going in circles." —Grant Stoelwinder

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1 Problem Statement

Consider a bead which is constrained to move along a frictionless, circular wire loop of radius R, subject to a gravitational acceleration $-g\hat{z}$ (where r is the cylindrical radial coordinate). In turn, the entire loop is made to rotate with a spin $\vec{\Omega} = \Omega\hat{z}$. Define the position of the bead to be at an angle θ relative to the bottom of the loop. Figure 1 shows a diagram of the setup.

(a) Find the values $\theta = \theta_0 \in (-\pi, +\pi]$ where the bead is in equilibrium.

(b) Determine whether the equilibrium points found in part (a) are stable or unstable.

(c) Find the frequencies of small-amplitude oscillations around the stable equilibria found in part (b).

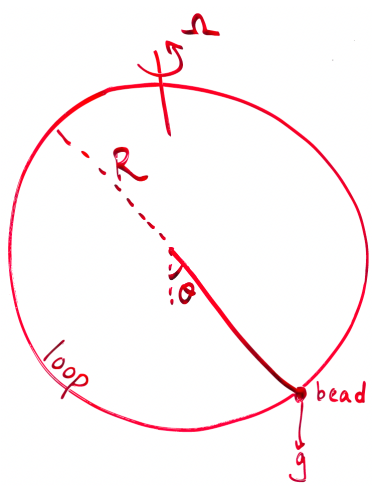

Figure 1: Diagram of the system

2 Problem Solution

In this problem, there are four quantities which have dimensions: g, R, Ω, and t (time). The physics will be controlled by the competition between the gravitational acceleration $\sim g$ and the acceleration due to the centrifugal force in the rotating frame $\sim R\Omega^2$. Therefore, for convenience, we define

$$\zeta = \frac{R\Omega^2}{g} \tag{1}$$

When $\zeta = 0$, the loop is not rotating at all. When $\zeta \to \infty$, the force of gravity is negligible.

The Coriolis force which arises in the rotating frame turns out to not matter, since it only acts in the direction of rotation (which is constrained by the wire).

2.1 Part (a): Equilibria

In this problem, there is a constraint force which the wire imposes upon the bead to keep it from leaving the wire. However, that constraint force (which is very hard to calculate) is zero in the direction along the wire. Therefore, we want to calculate the components of the gravitational and centrifugal forces *along* the wire. We can decompose these forces into the direction parallel to and perpendicular to the wire, as in the diagram in Figure 2.

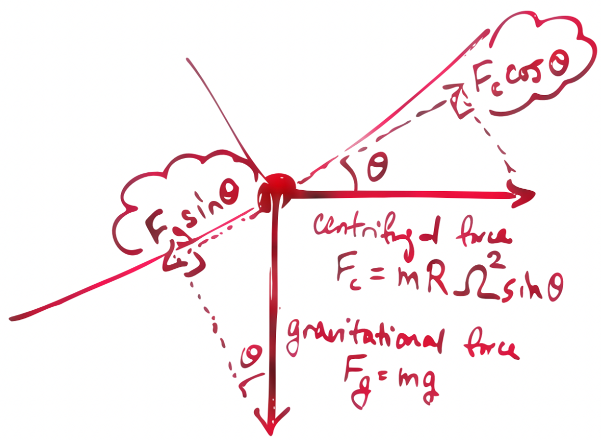

Figure 2: Decomposition of forces along the wire

We can write down Newton's second law along this direction,

$$F = ma \tag{2}$$

where m is the mass of the bead (which will turn out to cancel out).

In this case, the net force along the wire in the rotating frame is given by

$$F = mr\Omega^2 \cos\theta - mg\sin\theta = mR\Omega^2 \sin\theta\cos\theta - mg\sin\theta \tag{3}$$

The acceleration along the wire in this frame is given by $a = R\ddot{\theta}$, where $\ddot{\theta}$ is the second time derivative of θ. Then Newton's second law gives

$$\ddot{\theta} = -\frac{g}{R}\sin\theta + \frac{1}{2}\Omega^2 \sin 2\theta \equiv f \tag{4}$$

where we have divided out mR.

The equilibrium points $\theta = \theta_0$ are the angles at which a bead would not accelerate if placed exactly there. This is where $f = 0$, which gives

$$f = -\frac{g}{R}\sin\theta_0 + \Omega^2 \sin\theta_0 \cos\theta_0 = -\frac{g}{R}\sin\theta_0 \left(1 - \zeta\cos\theta_0\right) = 0 \tag{5}$$

We see that there are equilibria when either

$$\sin\theta_0 = 0 \tag{6}$$

or

$$\cos\theta_0 = \zeta^{-1} \tag{7}$$

- The first equation is solved by $\theta_0 = 0$ and $\theta_0 = \pi$, which always exist.
- The second equation is solved by $\theta_0 = \theta^*_\pm \equiv \pm \arccos\left(\zeta^{-1}\right)$, but these solutions only exist when $\zeta \geq 1$, because cosine can only take values between -1 and $+1$.

Note that when $\zeta = 1$ exactly, the second equation is only solved by $\theta_0 = 0$ (the same as given by the first equation).

Table 1 summarizes these solutions.

θ_0	when?
0	always
π	always
$\theta^*_+ = +\arccos\left(\zeta^{-1}\right)$	$\zeta > 1$
$\theta^*_- = -\arccos\left(\zeta^{-1}\right)$	$\zeta > 1$

Table 1: Equilibrium bead positions θ_0, and conditions for existence

The first equilibrium in Table 1 simply corresponds to the bead sitting at the bottom of the wire, and the second corresponds to the bead balancing perfectly on the top—these are forced to exist by the symmetry of the problem.

The third and fourth equilibria only exist when the centrifugal force $\sim R\Omega^2$ can compete with gravity $\sim g$. When $\zeta \to \infty$, these equilibria approach $\theta_0 \to \pm\pi$, as if there is no gravity at all.

Figure 3 shows a sketch of the equilibrium points as a function of ζ.

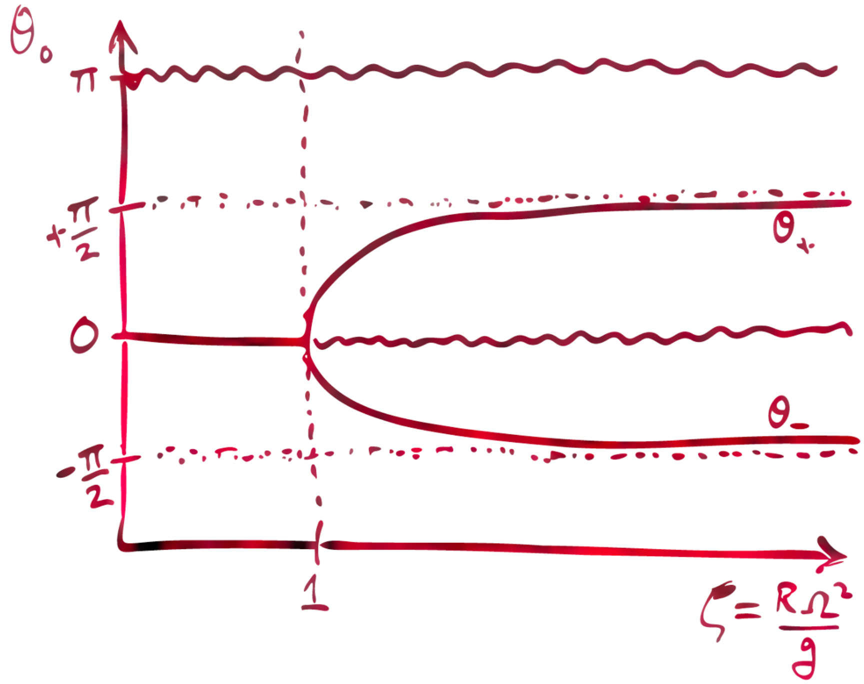

Figure 3: Stable (*straight*) and unstable (*squiggly*) equilibria θ_0 as a function of $\zeta = R\Omega^2/g$

2.2 Part (b): Stability

Suppose our particle is displaced very slightly from an equilibrium point to an angle

$$\theta = \theta_0 + \delta\theta \tag{8}$$

where $\delta\theta \ll 1$. Then

$$\delta\ddot{\theta} = \ddot{\theta} \tag{9}$$

and Newton's second law becomes

$$\ddot{\theta} = f(\theta_0) + \frac{\mathrm{d}f(\theta_0)}{\mathrm{d}\theta}\delta\theta + \mathcal{O}(\delta\theta^2) = \frac{\mathrm{d}f(\theta_0)}{\mathrm{d}\theta}\delta\theta + \mathcal{O}(\delta\theta^2) \tag{10}$$

where $f(\theta_0) = 0$ since θ_0 is an equilibrium point.

We recall that simple harmonic oscillation occurs in x when its equation of motion takes the form

$$\ddot{x} = -kx \tag{11}$$

for some $k > 0$. Therefore, for small amplitudes, stable equilibria occur when

$$\frac{\mathrm{d}f(\theta_0)}{\mathrm{d}\theta} < 0 \tag{12}$$

and, conversely, unstable equilibria occur at

$$\frac{\mathrm{d}f(\theta_0)}{\mathrm{d}\theta} > 0 \tag{13}$$

We can then compute the derivative of f with respect to θ:

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta} = -\frac{g}{R}\left(\cos\theta - 2\zeta\cos^2\theta + \zeta\right) \tag{14}$$

- For $\theta_0 = 0$, we have

  $$\frac{\mathrm{d}f(0)}{\mathrm{d}\theta} = -\frac{g}{R}(1-\zeta) \tag{15}$$

  We see that $\mathrm{d}f(0)/\mathrm{d}\theta$ is negative when $\zeta < 1$ and positive when $\zeta > 1$. Therefore, it is a stable equilibrium point when $\zeta < 1$, and but interestingly an *unstable* equilibrium point when $\zeta > 1$. When $\zeta = 1$ exactly, there is no "simple harmonic" oscillatory behavior below any amplitude around $\theta_0 = 0$.

- For $\theta_0 = \pi$, we have

  $$\frac{\mathrm{d}f(\pi)}{\mathrm{d}\theta} = +\frac{g}{R}(1+\zeta) > 0 \tag{16}$$

  Therefore, $\theta_0 = \pi$ is always an *unstable* equilibrium point, for any value of ζ.

- For $\theta_0 = \theta^*_\pm$, we have

$$\frac{\mathrm{d}f(\pi)}{\mathrm{d}\theta} = -\Omega^2\left(1-\zeta^{-2}\right) < 0 \tag{17}$$

  This is always negative as long as these equilibrium points exist ($\zeta > 1$).

Table 2 summarizes these results.

θ_0	stability ($\zeta < 1$)	stability ($\zeta > 1$)
0	stable	unstable
π	unstable	unstable
θ^*_+	–	stable
θ^*_-	–	stable

Table 2: Stability of equilibria

2.3 Part (c): Oscillation frequencies

In simple harmonic oscillation, the time dependence of $\delta\theta$ is oscillatory, i.e.,

$$\delta\theta \propto \cos(\omega t + \phi) \tag{18}$$

Then

$$\delta\ddot{\theta} = -\omega^2\delta\theta \tag{19}$$

and we therefore have

$$-\omega^2\delta\theta = \frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\delta\theta \tag{20}$$

We see that the oscillation frequencies are given by

$$\omega = \sqrt{-\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}} \tag{21}$$

Using the derivatives calculated in part (b), we can compute these frequencies:

- Around $\theta_0 = 0$, and for $\zeta < 1$,

$$\omega = \sqrt{-\frac{\mathrm{d}f(0)}{\mathrm{d}\theta}} = \sqrt{\frac{g}{R}(1-\zeta)} \tag{22}$$

  When $\zeta = 0$ (the wire is not spinning), we recognize ω to approach $\sqrt{g/R}$, the expression for the frequency of small oscillations of a pendulum of length R (which, of course, this problem becomes identical to). As the wire is made to spin faster, the bead would oscillate more and more slowly until gravity is no longer strong enough to keep the bead oscillating.

- Around $\theta_0 = \theta_\pm$, and for $\zeta > 1$,

$$\omega = \sqrt{-\frac{\mathrm{d}f(\theta_\pm)}{\mathrm{d}\theta}} = \Omega\sqrt{1-\zeta^{-2}} \tag{23}$$

  We see that, just above the critical ζ required for these equilibria to exist, oscillation is very slow. However, when ζ is very big, ω approaches the rotation frequency Ω, with vanishing sensitivity to gravity (which the bead no longer notices).

Table 3 summarizes these results.

θ_0	ω $(\zeta < 1)$	ω $(\zeta > 1)$
0	$\sqrt{(g/R)(1-\zeta)}$	–
π	–	–
θ_+^*	–	$\Omega\sqrt{1-\zeta^{-2}}$
θ_-^*	–	$\Omega\sqrt{1-\zeta^{-2}}$

Table 3: Frequencies of small-amplitude oscillations

As an aside, Equation 21 only produces real frequencies when $\mathrm{d}f(\theta_0)/\mathrm{d}\theta < 0$ (i.e., θ_0 is a stable equilibrium). When instead θ_0 is *unstable*, Equation 21 gives an imaginary frequency. Imaginary frequencies correspond to exponential growth or decay; in these cases, $\delta\theta$ will grow exponentially for small amplitudes.