California Institute of Technology
Physics 2a Recitation
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Duck Tales
26 October 2022

"You can't aim a duck to death." —Beverly Sills

Contents

1 Damped Harmonic Oscillators

Perhaps the most important differential equation in all of physics is the simple harmonic oscillator equation,

$$\dot{x} + \omega^2 x = 0 \tag{1}$$

which is solved by

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \tag{2}$$

for some constants A and B (to be determined by two initial conditions. Equation 1 describes energy-conserving oscillation in close vicinity to stable equilibria.

However, in realistic situations, there is often damping which is some function of $\dot{x}$. The damped harmonic oscillator incorporates a linear damping term $\propto \dot{x}$,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{3}$$

Here, the parameter γ has the same units as a frequency, and encodes the strength of the damping term. When $\gamma = 0$, Equation 3 is identical to Equation 1 with $\omega = \omega_0$, and thus its solution is known. In these Notes, we will fully solve Equation 3 for an arbitrary γ. Once we understand how Equation 3 behaves in general, we can pattern-match it to other damped harmonic equations which will naturally arise in many physics problems, find γ and ω_0, and instantly know the answer.

1.1 Transforming the damping away

We can try the following guess:

$$x = e^{-\gamma t} y \tag{4}$$

Here, we factor out an exponential decay and wish to write our equation in terms of $y(t)$. Then, upon taking some derivatives, we see that

$$\dot{x} = e^{-\gamma t}(\dot{y} - \gamma y) \tag{5a}$$

$$\ddot{x} = e^{-\gamma t}(\ddot{y} - 2\gamma\dot{y} + \gamma^2 y) \tag{5b}$$

Then we can substitute Equations and into Equation 3 to obtain

$$\ddot{y} - 2\gamma\dot{y} + \gamma^2 y + 2\gamma(\dot{y} - \gamma y) + \omega_0^2 y = 0 \tag{6}$$

where we have divided out the common factor of $e^{-\gamma t}$. This simplifies nicely to

$$\ddot{y} + (\omega_0^2 - \gamma^2)y = 0 \tag{7}$$

We notice then that Equation 7 (which is equivalent to Equation 3 using the substitution in Equation 1.1) is identical in form to the simple harmonic oscillator, Equation 1, for

$$\omega = \sqrt{\omega_0^2 - \gamma^2} \tag{8}$$

but only as long as $\omega_0 > \gamma$. Here, it becomes apparent that the relationship between ω_0 and γ in Equation 3 will make a big difference in the behavior of the solutions.

1.2 Underdamped oscillation

When $\omega_0 > \gamma$, Equation 7 really *is* a simple harmonic oscillator, and therefore we know that y has the solutions

$$y = A\cos(\omega t) + B\sin(\omega t) = A\cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right) + B\sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right) \tag{9}$$

so that the full damped harmonic equation has the solution

$$x = Ae^{-\gamma t}\cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right) + Be^{-\gamma t}\sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right) \tag{10}$$

We see that, in the underdamped case (Figure 1), the solutions are just oscillations (with modified frequency $\omega = \sqrt{\omega_0^2 - \gamma^2}$) modulated by an exponentially decaying envelope with damping rate γ.

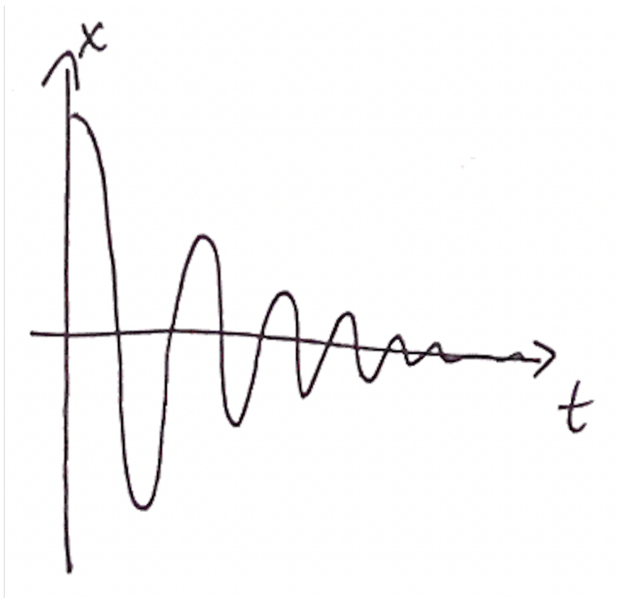

Figure 1: Underdamped oscillation.

1.3 Overdamped oscillation

When $\omega_0 < \gamma$, Equation 7 is no longer a simple harmonic oscillator. More specifically, in this case, we can define

$$\Gamma = \sqrt{\gamma^2 - \omega_0^2} \tag{11}$$

Then Equation 7 becomes

$$\ddot{y} = \Gamma^2 y \tag{12}$$

Equation 12 is not solved by oscillations, but rather growing and decaying exponentials—in fact, Equation 12 is what would be expected for dynamics in the vicinity of an *unstable* equilibrium:

$$y = Ae^{\Gamma t} + Be^{-\Gamma t} = Ae^{\sqrt{\gamma^2 - \omega_0^2}t} + Be^{-\sqrt{\gamma^2 - \omega_0^2}t} \tag{13}$$

Then the final solution (Figure 2) for x becomes

$$x = Ae^{-\gamma + \sqrt{\gamma^2 - \omega_0^2} t} + Be^{-\gamma - \sqrt{\gamma^2 - \omega_0^2} t} \tag{14}$$

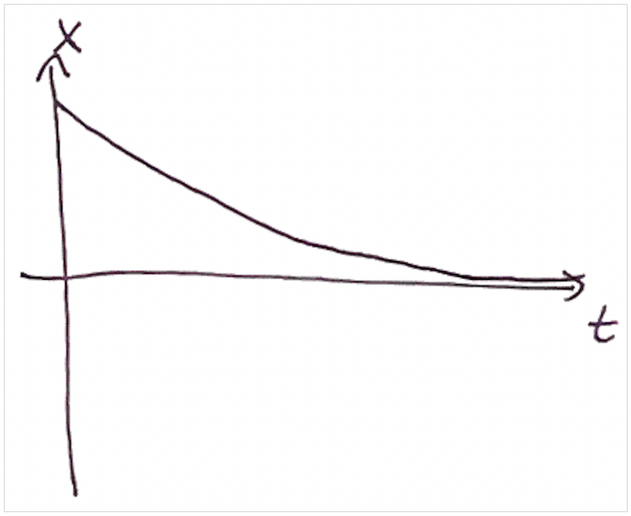

Figure 2: Overdamped oscillation.

We see that both of these terms are pure decays, with two different decay rates

$$\gamma_{\pm} = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{15}$$

We see that, while γ_+ becomes very large as the damping is increased, γ_- actually approaches zero when γ is large (for a fixed ω_0). Counterintuitively, then, increasing the damping constant γ in Equation 3 *does not* necessarily make a given solution decay faster.

1.4 Critically damped oscillation

The remaining case is $\omega_0 = \gamma$. This is the edge case between underdamped and overdamped oscillation. Equation 3 straightforwardly becomes

$$\ddot{y} = 0 \tag{16}$$

Thus, the solutions of y are simply those solutions for which the second derivative vanishes (i.e., lines):

$$y = At + B \tag{17}$$

Then

$$y = Ae^{-\gamma t} t + Be^{-\gamma t} \tag{18}$$

Critically damped oscillations (Figure 3) are exponentially decaying solutions with decay rate γ, possibly multiplied by t. In the absence of a specific solution, therefore, they guarantee the fastest decay of a solution.

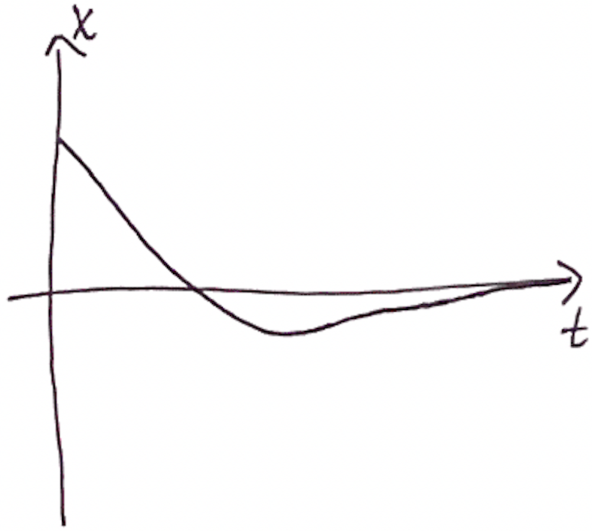

Figure 3: Critically damped oscillation.

2 Problem Statement

We first consider the case of a rubber duck which is allowed to bob up and down in a viscous liquid like honey (i.e., flow is of low Reynolds number). In this Problem, we will crudely model the duck as a uniformly dense, always vertical cylinder of mass m, radius R, and height H. Assume that the liquid has a density ρ and dynamic viscosity μ. The problem setup is shown in Figure 4.

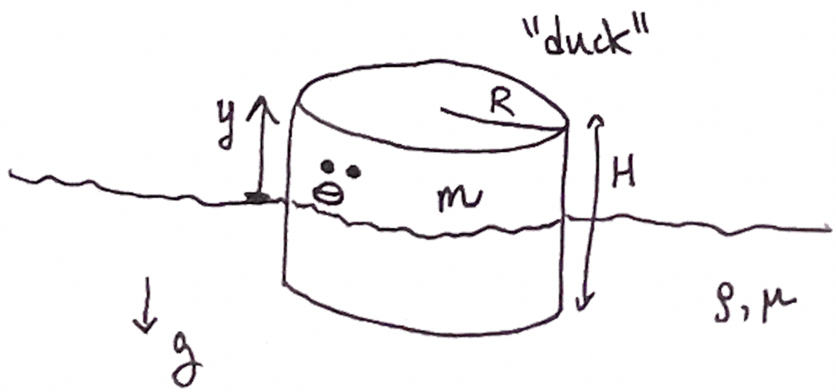

Figure 4: Diagram of a cylindrical duck submerged in a viscous liquid.

Besides the gravity of the duck, there is also a viscous drag force,

$$F_D = -bv = -6\pi R\mu v \tag{19}$$

where the coefficient is that of a sphere (as a crude approximation).

By Archimedes's principle, the buoyant force will act an upward force on the duck equal to the gravitational force of the amount of fluid that the duck has displaced.

(a) Define y to be the vertical coordinate of the duck, with positive y going upwards, and with $y = 0$ describing when the top of the duck cylinder exactly coincides with the surface of the liquid.

Use Newton's second law to write down a differential equation for y in terms of t.

For parts (a) and (b), you should leave the damping constant as b.

(b) Defining

$$y = y_0 + \delta y \tag{20}$$

transform the differential equation you found in part (a) to a differential equation for δy in the form of the damped harmonic oscillator equation, Equation 3, for a suitable y_0. When is this equation valid?

(c) By comparing your answer in part (b) and Equation 3, find ω_0 and γ.

(d) What is the condition on μ for the duck's oscillation to be critically damped? What is the general solution for δy in this case?

Now consider a non-driven, series RLC circuit, with resistance R, capacitance C, and inductance L (Figure 5).

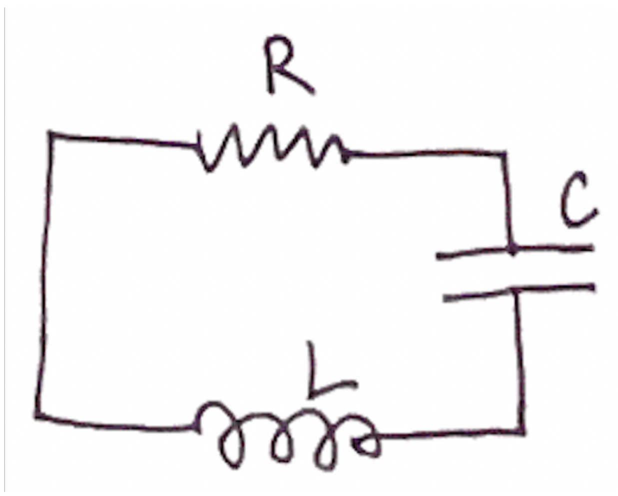

Figure 5: Series RLC circuit.

(e) What quantity is the same for every circuit element?

(f) What quantity, when summed up for every circuit element, must vanish?

(g) Use your answers to part (e) and (f) to find a differential equation for Q (the charge on the capacitor) of the form Equation 3.

(h) What is the condition on R for the oscillation to be underdamped? What is the general solution for Q in this case?

Now consider a non-driven, parallel RLC circuit, again with resistance R, capacitance C, and inductance L (Figure 6).

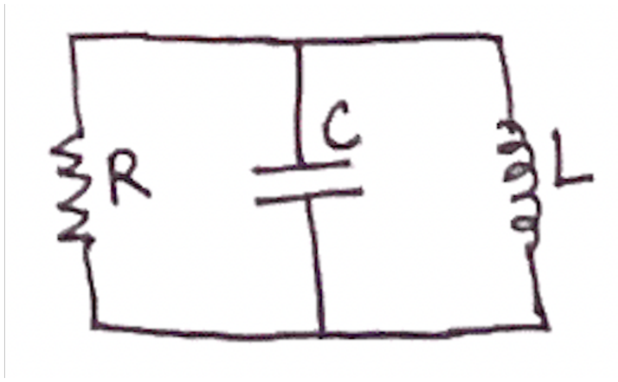

Figure 6: Parallel RLC circuit.

(i) What quantity is the same for every circuit element?

(j) What quantity, when summed up for every circuit element, must vanish?

(k) Use your answers to part (i) and (j) to find a differential equation for Q (the charge on the capacitor) of the form Equation 3.

(l) What is the condition on R for the oscillation to be overdampde? What is the general solution for Q in this case?

3 Problem Solutions

3.1 Part (a): Newton's second law for the duck

The gravitational force on the duck is

$$F_G = -mg \tag{21}$$

The buoyant force on the duck is equal to the amount of liquid mass displaced by the duck when it is at a height y, i.e.,

$$F_B = +\rho\pi R^2 g(H - y) \tag{22}$$

The drag force is

$$F_D = -bv \tag{23}$$

Then the net force is

$$F = -mg + \rho\pi R^2 g(H - y) - bv \tag{24}$$

By Newton's second law,

$$F = ma = -mg + \rho\pi R^2 g(H - y) - bv \tag{25}$$

This can be rewritten using $a = \ddot{y}$ and $v = \dot{y}$ as

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{\rho\pi R^2 g}{m} y = \frac{\rho\pi R^2 g H}{m} - g \tag{26}$$

3.2 Part (b): Removing the duck offset

Define

$$y = y_0 + \delta y \tag{27}$$

Then Equation 26 becomes

$$\delta\ddot{y} + \frac{b}{m}\delta\dot{y} + \frac{\rho\pi R^2 g}{m}\delta y + \frac{\rho\pi R^2 g}{m} y_0 = \frac{\rho\pi R^2 g H}{m} - g \tag{28}$$

Then we can cancel out the constant term from both sides by taking

$$\frac{y_0}{H} = 1 - \frac{m}{\rho\pi R^2 H} \tag{29}$$

Then

$$\delta\ddot{y} + \frac{b}{m}\delta\dot{y} + \frac{\rho\pi R^2 g}{m}\delta y = 0 \tag{30}$$

We see that this transformation is valid so long as $m < \rho\pi R^2 H$—otherwise, y_0 would lie at a height where the entire object is submerged, and in this case the volume that the submerged object actually no longer depends on height.

Since $\rho\pi R^2 H$ is just the mass of liquid that the duck cylinder would displace if it were fully submerged, this is equivalent to the condition that the duck must be less dense than the liquid (because otherwise it would not float).

3.3 Part (c): Duck analogy

By comparing Equation 30 to Equation 3, we have

$$\omega_0 = \sqrt{\frac{\rho\pi R^2 g}{m}} \tag{31a}$$

$$\gamma = \frac{b}{2m} \tag{31b}$$

3.4 Part (d): Critical duck damping

Critical duck damping occurs when

$$\omega_0 = \gamma \tag{32}$$

or, in this particular case,

$$\sqrt{\frac{\rho\pi R^2 g}{m}} = \frac{b}{2m} = \frac{6\pi R\mu}{2m} \tag{33}$$

This yields a "critical" dynamic viscosity

$$\mu = \sqrt{\frac{\rho m g}{9\pi}} \tag{34}$$

Note that this is distinct from the condition that the Reynolds number be low (i.e., viscosity dominates), which we have already assumed from the beginning.

The solution in the critically damped case is simply

$$\delta y = Ae^{-\gamma t}t + Be^{-\gamma t} = Ae^{-(b/2m)t}t + Be^{-(b/2m)t} \tag{35}$$

3.5 Part (e): Series *RLC* constraint 1

For a series circuit, the currents I_R, I_C, and I_L through the resistor, capacitor, and inductor, respectively, are all equal:

$$I_R = I_C = I_L \tag{36}$$

3.6 Part (f): Series *RLC* constraint 2

For a series circuit, the voltages V_R, V_C, and V_L across the resistor, capacitor, and inductor, respectively, must add up to zero (by Kirchhoff's current law):

$$V_R + V_C + V_L = 0 \tag{37}$$

3.7 Part (g): Series *RLC* differential equation

Equation 37 becomes

$$I_R R + \frac{1}{C}Q + L\dot{I}_L = 0 \tag{38}$$

However, note that $I_R = I_L = \dot{Q}$, so

$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = 0 \tag{39}$$

3.8 Part (h): Underdamped series *RLC*

Comparing to Equation 3, we identify

$$\omega_0 = 1/\sqrt{LC} \tag{40a}$$
$$\gamma = R/2L \tag{40b}$$
$$\tag{40c}$$

Then the condition for underdamped oscillation is

$$\omega_0 > \gamma \tag{41}$$

or, in this particular case,

$$\frac{1}{\sqrt{LC}} > \frac{R}{2L} \tag{42}$$

This requires an *upper limit* on R:

$$R < 2\sqrt{L/C} \tag{43}$$

In the underdamped case, the solution becomes

$$Q = Ae^{-\gamma t}\cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right) + Be^{-\gamma t}\sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right) \tag{44}$$

or, in this particular case,

$$Q = Ae^{-(R/2L)t}\cos\left(\sqrt{1/LC - (R/2L)^2}t\right) + Be^{-(R/2L)t}\sin\left(\sqrt{1/LC - (R/2L)^2}t\right) \tag{45}$$

3.9 Part (i): Parallel *RLC* constraint 1

For a parallel circuit, the currents V_R, V_C, and V_L across the resistor, capacitor, and inductor, respectively, are all equal:

$$V_R = V_C = V_L \tag{46}$$

3.10 Part (j): Parallel *RLC* constraint 2

For a series circuit, the currents I_R, I_C, and I_L through the resistor, capacitor, and inductor, respectively, must add up to zero (since the net current out of any point must vanish):

$$I_R + I_C + I_L = 0 \tag{47}$$

3.11 Part (k): Parallel *RLC* differential equation

We can first take the time derivative of Equation 47:

$$\frac{\mathrm{d}I_R}{\mathrm{d}t} + \frac{\mathrm{d}I_C}{\mathrm{d}t} + \frac{\mathrm{d}I_L}{\mathrm{d}t} = 0 \tag{48}$$

Then Equation 48 becomes

$$\frac{1}{R}\frac{\mathrm{d}V_R}{\mathrm{d}t} + \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{1}{L}V_L = 0 \tag{49}$$

However, $V_R = V_L = Q/C$, so

$$\ddot{Q} + \frac{1}{RC}\dot{Q} + \frac{1}{LC}Q = 0 \tag{50}$$

Note that the differential equation obtained here is almost identical to that of the series RLC circuit, except that the damping term actually scales with R^{-1} in the parallel case.

3.12 Part (l): Overdamped parallel RLC

Comparing to Equation 3, we identify

$$\omega_0 = 1/\sqrt{LC} \tag{51a}$$
$$\gamma = 1/2RC \tag{51b}$$

Then the condition for overdamped oscillation is

$$\omega_0 < \gamma \tag{52}$$

or, in this particular case,

$$\frac{1}{\sqrt{LC}} < \frac{1}{2RC} \tag{53}$$

This requires an *upper limit* on R:

$$R < \frac{1}{2}\sqrt{L/C} \tag{54}$$

Note that this is similar condition for overdamped oscillations in the parallel circuit is the same condition for underdamped oscillations as in the series circuit, since their damping terms depend on R in opposite ways.

In the overdamped case, the solution becomes

$$Q = Ae^{-\gamma+\sqrt{\gamma^2-\omega_0^2}t} + Be^{-\gamma-\sqrt{\gamma^2-\omega_0^2}t} \tag{55}$$

or, in this particular case,

$$Q = Ae^{-(1/2RC)+\sqrt{(1/2RC)^2-1/(LC)}t} + Be^{-(1/2RC)-\sqrt{(1/2RC)^2-1/(LC)}t} \tag{56}$$