California Institute of Technology
Physics 2a Recitation
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Practice Problem: “Supernatural Precession”
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“I’m so down to earth, I’m bringing gravity back.” —Tinie Tempah

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1 Problem Statement

1.1 Background

Newtonian mechanics predicts that the orbits of planets around the Sun will be exact ellipses, as long as the gravitational influences of the other planets are neglected. However, in reality, the planets (Mercury in particular, have been shown to have orbits which slowly precess over time. This means that, to very close approximation, they follow ellipses whose orientation changes over time (Figure 1). Roughly, this is because, in general relativity, gravity is "stronger" at close distances.

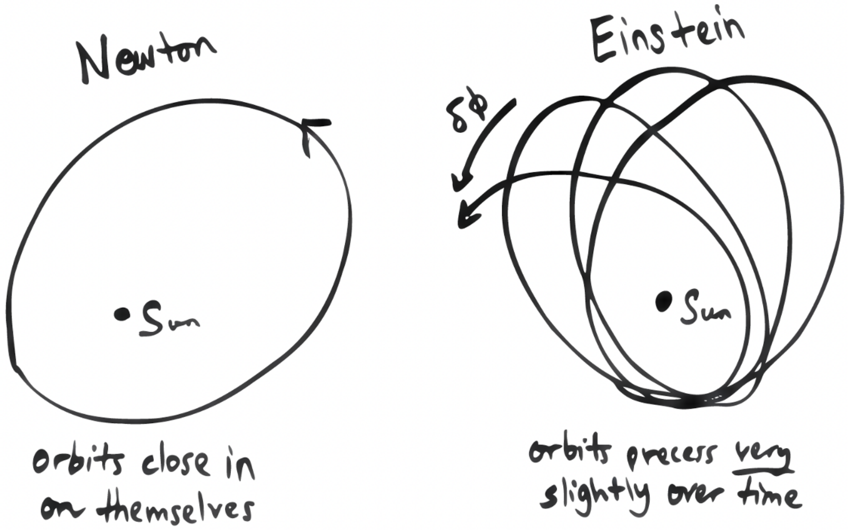

Figure 1: Newtonian closed orbits vs. general relativistic precessing orbits

It was commonly thought that the gravitational influence of other planets was causing this precession effect in Mercury's perihelion (i.e., angle of closest approach to the Sun). However, the observed value deviates from the Newtonian expectation quite significantly, by 43 arcsec century ^{1}, was too small to be explained by the known planets. This spurred a search for a new planet, "Vulcan," closer to the Sun than Mercury to explain this effect—however, Vulcan was never found.

Today, Mercury's perihelion precession is one of the canonical pieces of evidence for Einstein's theory of general relativity, which manifests as a subtle correction (but fundamental) to Newtonian gravity in our solar system.

In this problem, we will use our newfound understanding of small-amplitude simple harmonic oscillation to understand how the perihelion precession prediction arises out of general relativity.

We will draw strong analogies with the fact that, if some quantity x (with time derivative $\dot{x}$) obeys an "energy conservation" of the form

$$\frac{1}{2}\mu\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + V(x) = E \tag{1}$$

for some "mass" μ, "time" τ, and "energy" E, then x can oscillate (with small amplitudes) around the stable equilibrium points x_0 of $V(x)$, with "frequencies"

$$\omega = \sqrt{\frac{1}{m}\frac{\mathrm{d}^2 V(x_0)}{\mathrm{d}x^2}} \tag{2}$$

1.2 Problem

Let ϕ be the angle which describes the position of Mercury with respect to the Sun, and $r = r(\phi)$ be its distance from the Sun. *In this problem, ϕ will serve as the "time"—even though it isn't physically a time, it will still play the same role mathematically.*

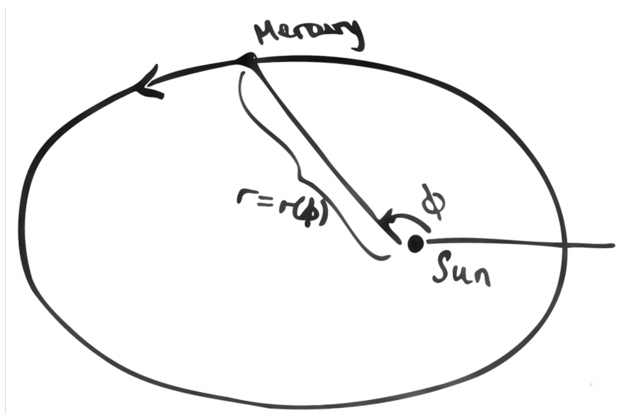

Figure 2: Diagram of the system, with ϕ and $r(\phi)$ labeled

Energy conservation in Newtonian mechanics gives a radial equation of the form

$$E = \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \left(\frac{L^2}{2mr^2} \quad \frac{GmM}{r}\right) \tag{3}$$

where E and L are the energy and angular momentum of the orbit respectively, and m and M are the masses of Mercury and the Sun, respectively ($m \quad M$). The frequency of the orbit (the time it takes ϕ to increase by 2π) is given by

$$\Omega = \sqrt{\frac{GM}{r_0^3}} \tag{4}$$

where r_0 is the semimajor axis of the orbit (for near-circular orbits, this is a stable equilibrium point). We can then rewrite Equation 3 using $\mathrm{d}\phi = \Omega\,\mathrm{d}t$:

$$E = \frac{1}{2}m\Omega^2\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \left(\frac{L^2}{2mr^2} \quad \frac{GmM}{r}\right) \tag{5}$$

where

$$U_{\mathrm{eff,c}}(r) = \frac{L^2}{2mr^2} \quad \frac{GmM}{r} \tag{6}$$

(a) Find the equilibrium point $r = r_0$ of $U_{\text{eff,c}}(r)$, and show that it is stable.

(b) Find the "frequency" η of small oscillations around this equilibrium point, i.e.,

$$\eta = \sqrt{\frac{1}{m\Omega^2} \frac{\mathrm{d}^2 U_{\text{eff,c}}(r_0)}{\mathrm{d}r^2}} \tag{7}$$

Note that the orbital angular momentum for circular orbits is given by

$$L = m\Omega r_0^2 \tag{8}$$

Note that this is not a frequency with respect to time, and won't have the same units. Instead, this is the rate at which the small radial oscillations occur, measured with respect to angle ϕ through the orbit.

(c) Argue that your result means that classical orbits do not precess.

$$0 = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + U_{\text{eff,gr}}(r) \tag{9}$$

where

$$U_{\text{eff,gr}}(r) = \frac{1}{2}\left[\left(1 \quad \frac{2GM}{rc^2}\right)\left(\frac{r^4 m^2 c^2}{L^2} + r^2\right) \quad \frac{E^2 r^4}{L^2 c^2}\right] \tag{10}$$

(d) In Equation 9, the "total energy" is zero. Use the fact that $\mathrm{d}r/\mathrm{d}\phi = 0$ at $r = r_0$ for circular orbits to solve for E in terms of r_0.

(e) Find the equilibrium points $r = r_0$ of $U_{\text{eff,gr}}(r)$. Use your answer from part (a) to eliminate E.

(f) Determine the stability of the equilibrium points that you find in part (a). Which of these corresponds to stable, circular orbits?

(g) Find the "frequency" η of small oscillations around the stable equilibrium point found in part (f).

Work only to first order in the small parameter

$$X \quad \frac{L^2 c^2}{G^2 m^2 M^2} \tag{11}$$

(h) Argue that the perihelion precession will be given by

$$\delta\phi \quad 2\pi(\eta \quad 1) \tag{12}$$

and evaluate the expression.

(i) Mercury's semimajor axis has been measured to be

$$r_0 \quad 0.387\,\text{a.u.} \quad 5.79 \quad 10^{10}\,\text{m} \tag{13}$$

and its orbital period has been measured to be

$$T = \frac{2\pi}{\Omega} \quad 88.0\,\text{d} \quad 7.60 \quad 10^{6}\,\text{s} \tag{14}$$

Furthermore

$$M = 1.99 \quad 10^{30}\,\text{kg} \tag{15}$$

What is the numerical value of $\delta\phi$, in arcsec century ^{1}? Does it agree with the observed result (Section 1.1)?

2 Problem Solution

2.1 Part (a): Newtonian equilibrium

The equilibrium point can be found by setting

$$\frac{\mathrm{d}U_{\text{eff,c}}(r_0)}{\mathrm{d}r} = 0 \tag{16}$$

Doing this gives

$$\frac{L^2}{mr_0^3} + \frac{GmM}{r_0^2} = 0 \tag{17}$$

Solving for r_0, we have

$$r_0 = \frac{L^2}{Gm^2M} \tag{18}$$

We can determine stability by taking the second derivative (at stable equilibria, the potential will be concave-up, i.e., $\mathrm{d}^2U_{\text{eff,c}}(r_0)/\mathrm{d}r^2 > 0$). This gives

$$\frac{\mathrm{d}^2U_{\text{eff,c}}(r_0)}{\mathrm{d}r^2} = \frac{3L^2}{mr_0^4} \quad \frac{2GmM}{r_0^3} = \frac{G^4m^7M^4}{L^6} = \frac{GmM}{r_0^3} > 0 \tag{19}$$

Therefore, r_0 is stable.

2.2 Part (b): Newtonian oscillation "frequency"

We can calculate the "frequency" η as

$$\eta = \sqrt{\frac{1}{m\Omega^2}\frac{\mathrm{d}^2U_{\text{eff,c}(r_0)}}{\mathrm{d}r^2}} = \sqrt{\frac{1}{m\Omega^2}\frac{GmM}{r_0^3}} = 1 \tag{20}$$

2.3 Part (c): No Newtonian precession

The "phase angle" of small radial oscillations is given by $\phi_r = \eta\phi$ (here, $\phi_r = \phi$ serves the same purpose as the time t in most other problems). When $\phi_r = \eta\phi$ increases by 2π, this means that Mercury has completed one *radial* oscillation.

However, since $\eta = 1$, this means that ϕ_r increases by 2π at the same time that ϕ increases by 2π (i.e., a full orbit). Therefore, we have shown (at least in this limit) that the orbit closes in on itself exactly, since r goes back to where it started exactly when the planet has undergone a single orbit.

It turns out that this result is true not just for small oscillations but also big ones. However, it is much more complicated to prove this.

2.4 Part (d): Eliminating *E*

Since $\mathrm{d}r/\mathrm{d}\phi = 0$ for a circular orbit, this means that

$$U_{\text{eff,gr}}(r_0) = \frac{1}{2}\left[\left(1 \quad \frac{2GM}{r_0c^2}\right)\left(\frac{r_0^4m^2c^2}{L^2} + r_0^2\right) \quad \frac{E^2r_0^4}{L^2c^2}\right] = 0 \tag{21}$$

Then

$$E = \sqrt{\left(1 \quad \frac{2GM}{r_0c^2}\right)\left(m^2c^4 + \frac{L^2c^2}{r_0^2}\right)} \tag{22}$$

2.5 Part (e): Relativistic equilibria

We can expand $U_{\text{eff,gr}}$ as

$$U_{\text{eff,gr}}(r) = \quad \frac{1}{2L^2c^2}\left(E^2 \quad m^2c^4\right)r^4 \quad \frac{Gm^2M}{L^2}r^3 + \frac{1}{2}r^2 \quad \frac{GM}{c^2}r \tag{23}$$

The equilibrium points occur where

$$\frac{\mathrm{d}U_{\text{eff,gr}}(r_0)}{\mathrm{d}r} = \quad \frac{2}{L^2c^2}\left(E^2 \quad m^2c^4\right)r_0^3 \quad \frac{3Gm^2M}{L^2}r_0^2 + r_0 \quad \frac{GM}{c^2} = 0 \tag{24}$$

We can plug in our expression for E from part (d) to obtain

$$r_0^2 \quad \frac{L^2}{Gm^2M}r_0 + \frac{3L^2}{m^2c^2} = 0 \tag{25}$$

Then, by the quadratic formula, we get two solutions for r_0:

$$r \quad = \frac{L^2}{2Gm^2M}\left(1 \quad \sqrt{1 \quad \frac{12G^2m^2M^2}{L^2c^2}}\right) \tag{26}$$

Note that these equilibria only exist when

$$1 \quad \frac{12G^2m^2M^2}{L^2c^2} \quad 0 \tag{27}$$

or

$$\frac{L}{m} \quad \frac{\sqrt{12}GM}{c^2} \tag{28}$$

At the critical case $L/m = \sqrt{12}GM/c^2$, we have

$$r \quad = r_{\rm ISCO} = \frac{6GM}{c^2} \tag{29}$$

Here, $r_{\rm ISCO}$ is the *innermost stable circular orbit.* We therefore have learned that, in general relativity, there is a *closest* stable circle orbit.

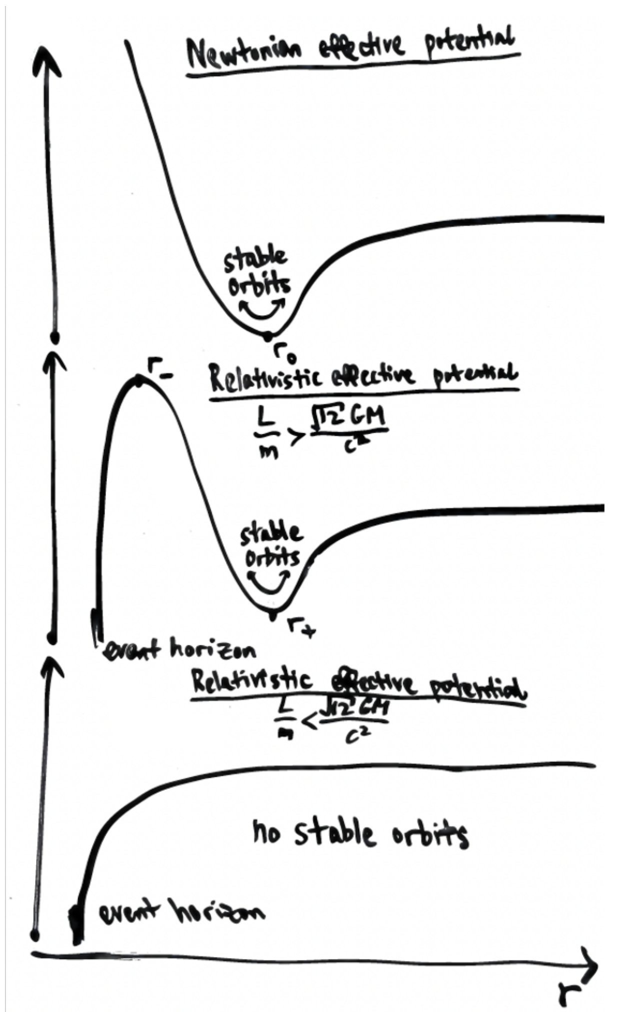

Figure 3: The classical effective potential $U_{\rm eff,c}(r)$ compared to the general relativistic effective potential $U_{\rm eff,gr}(r)/r^4$

2.6 Part (f): Relativistic orbit stability

To test stability, we must compute $\mathrm{d}^2 U_{\mathrm{eff,gr}(r)}/\mathrm{d}r^2$:

$$\frac{\mathrm{d}^2 U_{\mathrm{eff,gr}}(r)}{\mathrm{d}r^2} = \frac{6}{L^2c^2}\left(E^2 - m^2c^4\right) r^2 - \frac{6Gm^2M}{L^2} r_0 + 1 \tag{30}$$

Then, plugging in E, we have

$$\frac{\mathrm{d}^2 U_{\mathrm{eff,gr}}(r_0)}{\mathrm{d}r^2} = \frac{6Gm^2M}{L^2} r_0^{-1}\left(r_0^2 - \frac{5L^2}{6Gm^2M} r_0 + \frac{2L^2}{m^2c^2}\right) \tag{31}$$

The sign of $\mathrm{d}^2 U_{\mathrm{eff,gr}}(r_0)/\mathrm{d}r^2$ can therefore be seen to be the same as the signs of the (upward-opening) quadratic factor that appears in Equation 31.

Therefore, we see that an equilibrium point is stable when either

$$r_0 > \frac{5}{12}\frac{GM}{c^2} X^{-1}\left(1 + \sqrt{1 - \frac{288}{25}X}\right) \tag{32}$$

or

$$r_0 < \frac{5}{12}\frac{GM}{c^2} X^{-1}\left(1 - \sqrt{1 - \frac{288}{25}X}\right) \tag{33}$$

where we have defined $X \equiv G^2m^2M^2/L^2c^2 < 1/12$ (for distinct roots). Similarly, an equilibrium point is unstable when

$$\frac{5}{12}\frac{GM}{c^2} X^{-1}\left(1 + \sqrt{1 - \frac{288}{25}X}\right) < r_0 < \frac{5}{12}\frac{GM}{c^2} X^{-1}\left(1 + \sqrt{1 - \frac{288}{25}X}\right) \tag{34}$$

These conditions should be compared to

$$r_\pm = \frac{1}{2}\frac{GM}{c^2} X^{-1}\left(1 \pm \sqrt{1 - 12X}\right) \tag{35}$$

To show that r_+ is stable, it suffices to show that

$$\frac{1}{2} + \frac{1}{2}\sqrt{1 - 12X} > \frac{5}{12} + \frac{5}{12}\sqrt{1 - \frac{288}{25}X} \tag{36}$$

We can rearrange this to

$$\sqrt{1 - 12X} > -\frac{1}{6} + \frac{5}{6}\sqrt{1 - \frac{288}{25}X} \tag{37}$$

We see that these expressions both vanish at $X = 1/12$. Moreover, it can be seen that the left hand side has an everywhere-positive derivative, whereas the one on the right has an everywhere-negative derivative; therefore, the left hand side is always larger than the right, as desired.

To show that r_- is unstable, it suffices to show that

$$\begin{aligned} \frac{1}{2} - \frac{1}{2}\sqrt{1 - 12X} &< \frac{5}{12} + \frac{5}{12}\sqrt{1 - \frac{288}{25}X} \\ \frac{1}{2} - \frac{1}{2}\sqrt{1 - 12X} &> \frac{5}{12} - \frac{5}{12}\sqrt{1 - \frac{288}{25}X} \end{aligned} \tag{38}$$

Once again, these conditions can be manipulated to

$$\begin{aligned} \sqrt{1 - 12X} &> \frac{1}{6} - \frac{5}{6}\sqrt{1 - \frac{288}{25}X} \\ \sqrt{1 - 12X} &< \frac{1}{6} + \frac{5}{6}\sqrt{1 - \frac{288}{25}X} \end{aligned} \tag{39}$$

For the first condition, we see that the left hand side is everywhere non-negative, but the right hand side is always negative, since it is negative at $X = 0$ and is also always decreasing. For the second condition, we see that the left and right hand sides both equal 1 at $X = 0$. The former has a derivative of $-24/\sqrt{16 - 192X}$ and the latter $-24/\sqrt{25 - 288X}$. Because $16 - 198X < 25 - 288X$ for all $X > 0$, we see that, for $X > 0$, the left hand side always decreases faster (and is thus always less in value) than the right.

Therefore, r_+ is stable, and r_- is unstable.

2.7 Part (g): Relativistic oscillation "frequency"

We seek the radial "frequency" η for the stable equilibrium point $r = r_+$. When X is small, the binomial approximation can be used to write

$$r_+ = \frac{GM}{c^2}X^{-1}\left(1 - 3X + O(X^2)\right) \tag{40}$$

We can then plug in r_+ and keep only lowest order terms. This messy substitution eventually yields

$$\frac{\mathrm{d}^2 U_{\mathrm{eff,gr}}(r_+)}{\mathrm{d}r^2} = 1 + 12X + O(X^2) \tag{41}$$

Then

$$\eta = \sqrt{\frac{\mathrm{d}^2 U_{\mathrm{eff,gr}}(r_+)}{\mathrm{d}r^2}} = 1 + 6X \tag{42}$$

2.8 Part (h): Relativistic precession

When ϕ advances from 0 to 2π, the phase of small radial oscillations ϕ_r advances from 0 to $2\pi\eta$. Therefore, each orbit, ϕ_r slightly exceeds a full radial oscillation 2π by

$$\delta\phi \approx 2\pi(\eta - 1) = 6\pi X \tag{43}$$

corresponding to the precession rate (angle accumulated per orbit).

Using $L = m\Omega r_0^2$ and the classical Keplerian frequency $\Omega = \sqrt{GM/r_0^3}$, this can be written

$$\delta\phi \approx \frac{6\pi G^2 m^2 M^2}{L^2 c^2} = \frac{6\pi GM}{r_0 c^2} \tag{44}$$

2.9 Part (i): Observational prediction

Plugging in numbers, we have

$$\delta\phi \quad 4.81 \quad 10^{\ 7}\,\mathrm{rad} \quad 9.92 \quad 10^{\ 2}\,\mathrm{arcsec} \tag{45}$$

In a century, mercury undergoes approximately $100\,\mathrm{yr}/T \quad 415$ orbits. Then the total phase accumulated in a century is

$$\delta\phi_{\mathrm{century}} \quad 415\delta\phi \quad 41.2\,\mathrm{arcsec} \tag{46}$$

This is in pretty good agreement with the observed deviation from the Newtonian prediction 43 arcsec.

Congrats, Einstein.