California Institute of Technology
Physics 2a Recitation **Swing Dance**
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"To fear is one thing. To let fear grab you by the tail and swing you around is another." — Katherine Paterson

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1 Problem Statement

Consider a pendulum with length R with a mass M at the end—its position is described by the angle θ from the vertical. Attached to it is another pendulum with length r and mass m—its position is described by the angle ϕ from the vertical. The setup is shown in Figure 1.

In this Problem, we will assume that $\theta, \phi \ll 1$, i.e., we will adopt the small angle approximation ($\sin\theta \approx \tan\theta \approx \theta$, $\sin\phi \approx \tan\phi \approx \phi$, and $\cos\theta \approx \cos\phi \approx 1$).

This Problem is meant to be a very crude toy model of a playground swing set, and will help us understand why we must swing our legs the way that we do to use a swing. The quantities M and R are meant to represent your body's mass and the length of the swing, respectively, and the quantities m and r are meant to model the mass and length of your legs, respectively.

In reality, swings may not be at small angles, and the way you move your legs when you are on a swing are often also not at small angles. The Estonian sport of kiiking is a form of extreme swinging (on a physical pendulum) with the goal of swinging all the way around. However, this toy model has enough physics to capture a lot of interesting features about swinging.

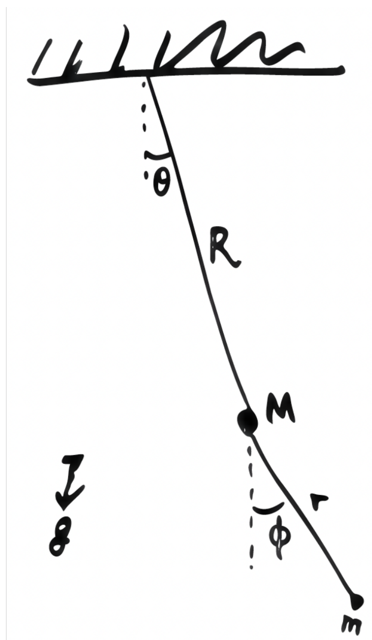

Figure 1: A diagram of a double pendulum, where the lower pendulum is driven. This is a very simple toy model for a child on a swing.

We first assume that there is no friction.

(a) Write down the total angular momentum L of the system about the axle of the upper pendulum.

(b) Write down the total torque acting on the system.

(c) Find an equation which relates θ, $\ddot{\theta}$, ϕ, and ϕ, using

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau \tag{1}$$

(d) Now assume that ϕ is driven at a frequency (i.e., $\phi = \phi_0 \cos(\omega t + \psi)$). Then the steady solution for θ is $\theta = \theta_0 e^{i\omega t}$.

Find an expression for θ_0/ϕ_0 in terms of ω. At which ω does (the magnitude of) this function peak? At which ω does it vanish?

(e) Now repeat problems (a)–(d), but assume that there is now a frictional torque on the top axle of the pendulum $-\lambda\dot{\theta}$.

This time, it is wise to add a complex driving $\phi = \phi_0 e^{i\omega t}$, and guess a solution $\theta = \theta_0 e^{i\omega t}$.

This is not necessarily the best guess for a frictional torque, but other expressions are not necessarily linear, and this is probably good enough.

2 Problem Solution

2.1 Part (a): Total angular momentum

The angular momentum is given by the linear momentum perpendicular to the line from the axle. Because of the small angle approximation, the momentum due to the masses oscillating is *all* in the perpendicular direction, and the distance of each mass away from the axle does not depend on the angle.

Then we can write down the angular momenta of each mass separately:

$$L_1 = MR^2\dot{\theta} \tag{2a}$$
$$L_2 = m(R\dot{\theta} + r\dot{\phi})(R + r) \tag{2b}$$

Then the total angular momentum is

$$L = L_1 + L_2 = MR^2\dot{\theta} + mR(R+r)\dot{\theta} + m(R+r)r\dot{\phi} = \left[(M+m)R^2 + mRr\right]\dot{\theta} + m(R+r)r\dot{\phi} \tag{3}$$

2.2 Part (b): Total torque

The torque (the perpendicular force multiplied by distance to the axle) is purely due to gravity on the two masses. We can write down the torques due to the two masses separately:

$$\tau_1 = -MgR\theta \tag{4a}$$
$$\tau_2 = -mg(R\theta + r\phi) \tag{4b}$$

Therefore, the total torque is

$$\tau = \tau_1 + \tau_2 = -MgR\theta - mg(R\theta + r\phi) = -(M+m)gR\theta - mgr\phi \tag{5}$$

2.3 Part (c): Equation for $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$

We can take the time derivative of L:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left[(M+m)R^2 + mRr\right]\ddot{\theta} + m(R+r)r\ddot{\phi} \tag{6}$$

However, this is equal to the total torque τ, so

$$\left[(M+m)R^2 + mRr\right]\ddot{\theta} + m(R+r)r\ddot{\phi} = -(M+m)gR\theta - mgr\phi \tag{7}$$

We can arrange this equation so that θ and $\ddot{\theta}$ are on one side, and ϕ and $\ddot{\phi}$ are on the other:

$$\left[(M+m)R^2 + mRr\right]\ddot{\theta} + (M+m)gR\theta = -\left[m(R+r)r\ddot{\phi} + mgr\phi\right] \tag{8}$$

We recognize this to look like a harmonic oscillator equation in θ on one side, but equated to a similar harmonic oscillator equation in ϕ on the other.

2.4 Part (d): Harmonic solutions

We force $\phi = \phi_0 \cos(\omega t + \psi)$. This sort of forcing cannot change the total angular momentum (since it comes from within the system) so our equations still apply.

We notice that

$$\ddot{\phi} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\phi_0 \cos(\omega t + \psi)\right) = -\omega^2 \phi_0 \cos(\omega t + \psi) = -\omega^2 \phi \tag{9}$$

Then, plugging this in, we have

$$\left[(M+m)R^2 + mRr\right]\ddot{\theta} + (M+m)gR\theta = \left[m(R+r)r\omega^2 - mgr\right]\phi_0 \cos(\omega t + \psi) \tag{10}$$

Then we guess $\theta = \theta_0 \cos(\omega t + \psi)$, so that

$$-\left[\left[(M+m)R^2 + mRr\right]\omega^2 - (M+m)gR\right]\theta_0 \cos(\omega t + \psi) = \left[m(R+r)r\omega^2 - mgr\right]\phi_0 \cos(\omega t + \psi) \tag{11}$$

We see that, because of our guesses, $\cos(\omega t + \psi)$ cancels out on both sides. We can then rearrange Equation 11 as

$$\frac{\theta_0}{\phi_0} = -\frac{m(R+r)r\omega^2 - mgr}{\left[(M+m)R^2 + mRr\right]\omega^2 - (M+m)gR} \tag{12}$$

This function's magnitude diverges at the resonance (where the denominator goes to zero):

$$\omega_* = \sqrt{\frac{g}{R + mr/(M+m)}} \tag{13}$$

The function itself vanishes at

$$\omega_0 = \sqrt{\frac{g}{R+r}} \tag{14}$$

We can notice that $\omega_0 < \omega_*$ always, except when $m = 0$ or $r = 0$ (i.e., the lower pendulum does not exist).

We can also define the ratios $\mu = m/M$ and $\rho = r/R$ (comparing the masses and pendulum lengths). Then Equation 12 can be rewritten as

$$\frac{\theta_0}{\phi_0} = -\frac{\mu\rho + \rho^2}{1 + \mu + \mu\rho} \frac{\omega^2 - \omega_0^2}{\omega^2 - \omega_*^2} \tag{15}$$

A sketch of θ_0/ϕ_0 is shown in Figure 2. While it is clear that there is some characteristic frequency where driving from the lower pendulum causes the upper pendulum to move (note that the frequencies are $\approx \sqrt{g/R}$), it is not so clear what is happening at the singularity. It does seem, however, that driving which occurs right below the resonance causes oscillations which are *in phase* with the driving, whereas driving either slightly above or far away from the resonance in either direction causes oscillations in the upper pendulum which are *out of phase* with the driving.

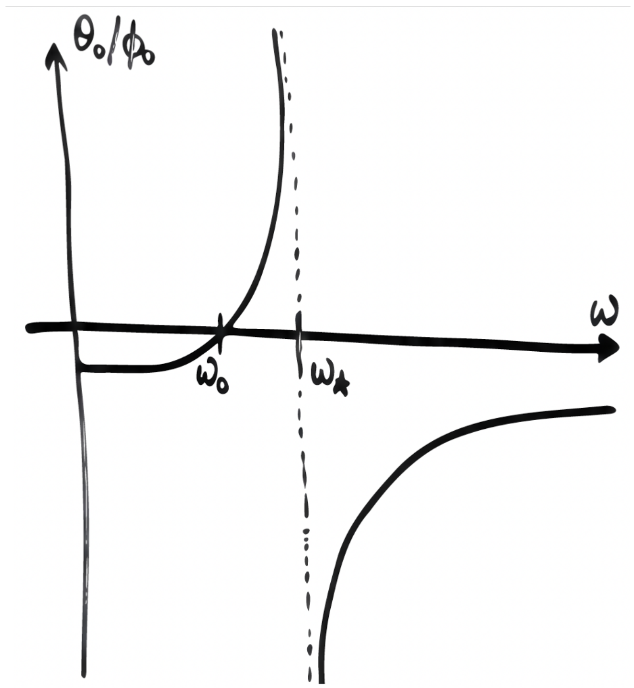

Figure 2: The function θ_0/ϕ_0 plotted against ω.

2.5 Part (e): Damped solutions

The expression for the total angular momentum L is still the same, but now the torque has an additional damping term:

$$\tau = -(M + m)gR\theta - mgr\phi - \lambda\dot{\theta} \tag{16}$$

Then the equation $\tau = \mathrm{d}L/\mathrm{d}t$ becomes

$$\left[(M + m)R^2 + mRr\right]\ddot{\theta} + \lambda\dot{\theta} + (M + m)gR\theta = -\left[m(R + r)r\ddot{\phi} + mgr\phi\right] \tag{17}$$

Before, we were able to guess solutions $\propto \cos(\omega t + \psi)$ because the second derivative of cosine only multiplies it by $-\omega^2$ (in other words, cosine is an "eigenfunction" of $\mathrm{d}^2/\mathrm{d}t^2$). However, acting a first derivative $\mathrm{d}/\mathrm{d}t$ turns $\cos(\omega t + \psi)$ into $\propto \sin(\omega t + \psi)$ instead of just multiplying it by a simple constant, so it is inconvenient now to guess it (i.e., cosine is *not* an eigenfunction of $\mathrm{d}/\mathrm{d}t$).

However, complex exponentials still encode oscillations, while also acting simply under all numbers of derivatives: $\mathrm{d}e^{i\omega t}/\mathrm{d}t = i\omega e^{i\omega t}$ and $\mathrm{d}^2 e^{i\omega t}/\mathrm{d}t^2 = -\omega^2 e^{i\omega t}$. Therefore, it is good to guess them (and take the real part at the end to obtain the actual desired solution).

Plugging in a complex exponential thus replaces time derivatives with $i\omega$:

$$-\left[\omega^2\left[(M+m)R^2 + mRr\right] - i\omega\lambda - (M+m)gR\right]\theta_0 = \left[m(R+r)r\omega^2 - mgr\right]\phi_0 \tag{18}$$

Then, defining

$$\gamma = \frac{\lambda}{(M+m)R^2 + mRr} \tag{19}$$

Then

$$\frac{\theta_0}{\phi_0} = -\frac{\mu\rho + \rho^2}{1+\mu+\mu\rho}\frac{\omega^2 - \omega_0^2}{\omega^2 - i\gamma\omega - \omega_*^2} \tag{20}$$

Figure 3 shows a sketch of the magnitude of θ_0/ϕ_0, as well as the phase. In the dissipationless case, the phase was always either 0 or π (corresponding to whether θ_0/ϕ_0 was positive or negative), but now, when the system is driven exactly at resonance $\omega = \omega_*$, we see that

$$\frac{\theta_0}{\phi_0} = -i\frac{\mu\rho + \rho^2}{1+\mu+\mu\rho}\frac{\omega_*^2 - \omega_0^2}{\gamma\omega_*} \tag{21}$$

In other words, if $\phi = |\phi_0|e^{i\omega t}$ (so that its real part is $\cos(\omega t)$), then $\theta = \theta_0 e^{i\omega t} = |\theta_0|(-i)e^{i\omega t} = |\theta_0|e^{i(\omega t - \pi/2)}$.

That means that, at resonance, the upper pendulum (i.e., the swing) lags behind the motion of the lower pendulum (i.e., your legs). If you reflect on how you instinctively move your legs on a swing, you will realize that your legs will be extended when the swing is at its lowest, and only when your legs begin to turn the other way does the swing reach its maximum—this is a manifestation of this phase shift which is predicted by our toy model.

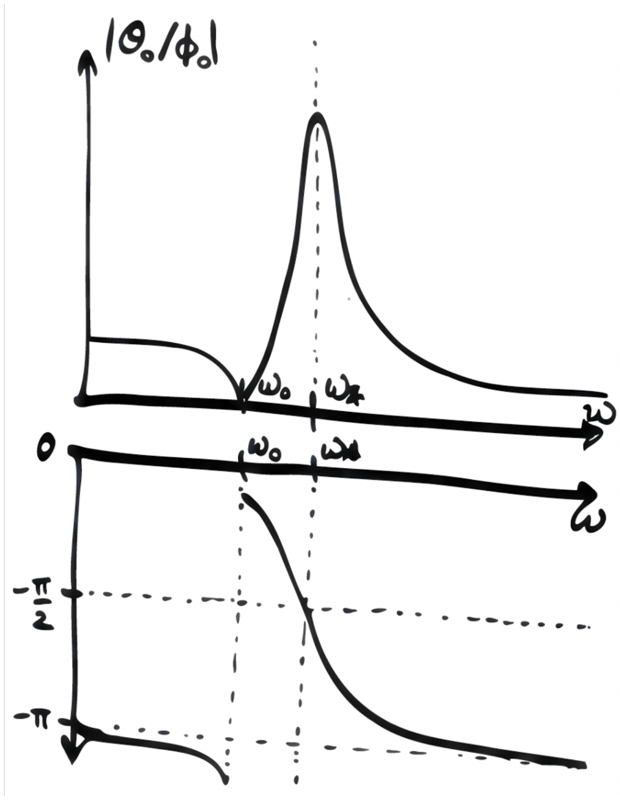

Figure 3: The magnitude (*top*) and argument (*bottom*) of θ_0/ϕ_0. The former corresponds to how intensely the upper pendulum will be made to oscillate, and the latter corresponds to the phase difference between the driving and the oscillation of the upper pendulum.