California Institute of Technology
Physics 2a Recitation
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Use the Central Force
28 October 2022

"Airplanes are the poor man's satellites." —Steven Magee

Contents

1 Problem Statement

The total energy of a system orbiting in a plane in a central potential $U(r)$ is

$$E = \frac{1}{2}m|\vec{v}|^2 + U(r) = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\phi^2 + U(r) \tag{1}$$

A central force (i.e., spherically symmetric) potential cannot change the angular momentum of the system (about $r = 0$). The angular momentum is given by

$$L = mv_\phi r \tag{2}$$

Therefore, E can be rewritten as

$$E = \frac{1}{2}mv_r^2 + U_{\text{eff}}(r) \tag{3}$$

where

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) \tag{4}$$

is an **effective potential** now including a **centrifugal barrier** term—Equation 3 now describes the total energy for a *one-dimensional* problem which is equivalent to the full two-dimensional one.

In this Problem, we will consider a general power law potential

$$U(r) = \alpha r^\beta \tag{5}$$

for some constants α and β.

(a) Find the equilibrium point(s) $r = r_0$ of $U_{\text{eff}}(r)$. Find the conditions on α and β for it to exist (note that $r_0 > 0$).

(b) When are the equilibrium point(s) found in part (a) stable?

(c) When the equilibrium point(s) found in part (a) is stable, what is the frequency κ (the **epicyclic frequency**) of small, radial oscillations?

(d) Note that $v_\phi = r\Omega$. What is κ/Ω? When is this ratio either 1 or 2, and what is the significance of this?

In the absence of matter (e.g., outside of a central mass), the flux of the gravitational field is conserved. Poisson's equation states that the flux Φ around a central mass M is

$$\Phi = \int \vec{g} \cdot \mathrm{d}\vec{A} = -4\pi GM \tag{6}$$

Since the surface area of a sphere at a radius r is $4\pi r$, and since the gravitational acceleration is the same everywhere on this surface,

$$g_r(4\pi r^2) = -4\pi GM \tag{7}$$

Therefore, in three (spatial) dimensions,

$$g_r = -\frac{GM}{r^2} \tag{8}$$

and the gravitational potential of a mass m is

$$U(r) = \int_{\infty}^{r} m g_r(r')\,\mathrm{d}r' = -\frac{GmM}{r} \tag{9}$$

(e) For motion around a massive point source in three-dimensional gravity, what are α and β? Are there stable circular orbits?

(f) Consider two-dimensional gravity. Are there stable circular orbits?

(g) Consider four-dimensional gravity. Are there stable circular orbits?

Observations reveal that most of the Milky Way galaxy (outside of the "bulge" of the galaxy) has a "flat rotation curve," i.e., the orbital velocity v_ϕ around the galaxy is a constant with radius. This means that

$$\Omega = v_\phi / r \propto r^{-1} \tag{10}$$

(h) The gravitational force due to an extended, spherically symmetric mass distribution is given by

$$F_r = -\frac{GmM(r)}{r^2} \tag{11}$$

where $M(r)$ is the amount of mass enclosed within a radius r.

Find how $M(r)$ must scale with r to produce the observed flat rotation curve.

The mass of the Milky Way is primarily concentrated within the bulge—objects far outside of the bulge thus enclose a radius-independent (constant with radius) amount of visible mass. Is this compatible with your answer for $M(r)$?

2 Problem Solutions

2.1 Part (a): Equilibria

The equilibria of $U_{\text{eff}}(r)$ are where its derivative vanishes:

$$\left.\frac{\mathrm{d}U_{\text{eff}}(r)}{\mathrm{d}r}\right|_{r=r_0} = -\frac{L^2}{mr_0^3} + \alpha\beta r_0^{\beta-1} = \frac{1}{mr_0^3}\left(m\alpha\beta r_0^{\beta+2} - L^2\right) = 0 \tag{12}$$

The equilibrium point occurs when

$$L^2 = m\alpha\beta r_0^{\beta+2} \tag{13}$$

We see that this only has a solution when $\alpha\beta > 0$. For $\beta \neq 0, -2$, there is a single equilibrium point is at

$$r_0 = \left(\frac{L^2}{m\alpha\beta}\right)^{1/(\beta+2)} \tag{14}$$

For $\beta = -2$, every r is an equilibrium point so long as $L^2 = m\alpha\beta$. Otherwise, there are no equilibrium points.

For $\beta = 0$, there are no equilibrium points at all—this corresponds to a constant potential $U(r) =$ const., which cannot overcome the centrifugal barrier to make an equilibrium point.

2.2 Part (b): Stability

Stability of the equilibrium point r_0 found in part (a) is determined by the second derivative of $U_{\text{eff}}(r)$:

$$\left.\frac{\mathrm{d}^2 U_{\text{eff}}(r)}{\mathrm{d}r^2}\right|_{r=r_0} = \frac{3L^2}{mr_0^4} + \alpha\beta(\beta-1)r_0^{\beta-2} = r_0^{\beta-2}\alpha\beta(\beta+2) \tag{15}$$

We see that stability will occur so long as $\beta > -2$ (note that $\alpha\beta$ is positive). For $\beta = -2$, when L is such that there are equilibria, they are at all r (i.e., $U_{\text{eff}}(r)$ is flat), and thus there is no stable (or unstable) equilibrium anywhere.

2.3 Part (c): Epicyclic frequency

The frequency κ of small-amplitude radial oscillations is

$$\kappa = \sqrt{\left.\frac{1}{m}\frac{\mathrm{d}^2 U_{\text{eff}}(r)}{\mathrm{d}r^2}\right|_{r=r_0}} = r_0^{\beta/2-1}\sqrt{m^{-1}\alpha\beta(\beta+2)} \tag{16}$$

2.4 Part (d): Ratio of epicyclic and orbital frequencies

The orbital frequency Ω for circular orbits at r_0 is given by

$$\Omega = \sqrt{\frac{L}{mr_0^2}} = r_0^{\beta/2-1}\sqrt{m^{-1}\alpha\beta} \tag{17}$$

Therefore,

$$\frac{\kappa}{\Omega} = \sqrt{\beta+2} \tag{18}$$

We see that $\kappa/\Omega = 1$ when $\beta = -1$ (gravity around a point mass), and $\kappa/\Omega = 2$ when $\beta = 2$ (Hooke's law). These are the only central force potential exponents which admit elliptical orbits, those for which the orbit closes without crossing itself (Figure 1). For the former, $r = 0$ lies at a focus of the ellipse. For the latter, $r = 0$ lies at the center of the ellipse.

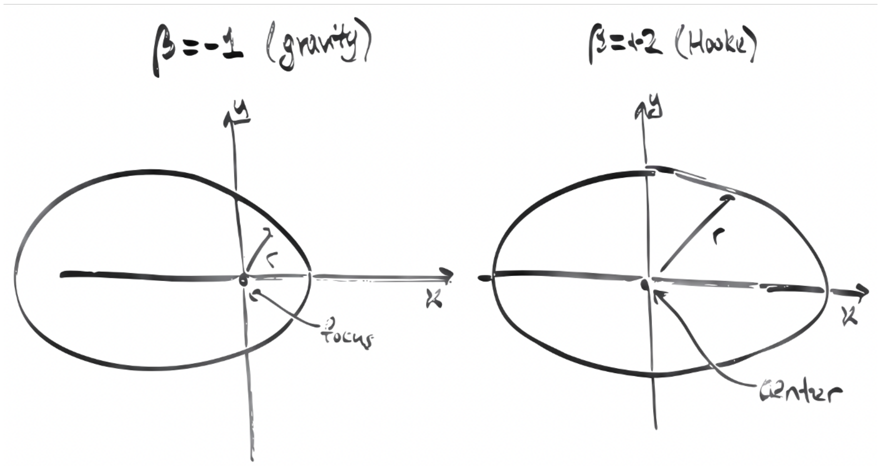

Figure 1: Elliptical orbits for $\beta = -1$ (gravity) and $\beta = +2$ (harmonic oscillator).

2.5 Part (e): Three-dimensional gravity

In three-dimensional gravity, we can compare

$$U(r) = -\frac{GmM}{r} = \alpha r^{\beta} \tag{19}$$

We see that

$$\alpha = -GmM \tag{20a}$$
$$\beta = -1 \tag{20b}$$

Since $\alpha\beta > 0$ and $\beta \neq 0, -2$, we have stable orbits.

2.6 Part (f): Two-dimensional gravity

For two-dimensional gravity, the (constant) gravitational flux $-4\pi GM$ (for suitably defined G) needs to be spread over a circumference $2\pi r$ (rather than a surface area $4\pi r^2$, in the three-dimensional case). The gravitational force F_r thus scales as

$$F_r = -\frac{2GmM}{r} \propto r^{-1} \tag{21}$$

This yields a potential

$$U(r) = 2GmM \ln(r/r_*) \tag{22}$$

where r_* is arbitrary. This is *not* a power law potential, so there is no direct analogy for α and β.

However, there exist circular orbits when

$$\left.\frac{\mathrm{d}U_{\text{eff}}(r)}{\mathrm{d}r}\right|_{r=r_0} = -\frac{L^2}{mr_0^3} + \frac{2GmM}{r_0} = 0 \tag{23}$$

This occurs when

$$r_0 = \sqrt{\frac{L^2}{2Gm^2M}} \tag{24}$$

or

$$L^2 = 2Gm^2Mr_0^2 \tag{25}$$

We can verify that they are stable using the second derivative of $U_{\text{eff}}(r)$:

$$\left.\frac{\mathrm{d}^2U_{\text{eff}}(r)}{\mathrm{d}r^2}\right|_{r=r_0} = \frac{3L^2}{mr_0^4} - \frac{2GmM}{r_0^2} = \frac{4GmM}{r_0^2} > 0 \tag{26}$$

Interestingly, even though two-dimensional gravity also has stable circular orbits, there is no such thing as an escape velocity—as can be seen from Equation 22, there is no initial kinetic energy big enough to overwhelm the two-dimensional gravitational potential at $r \to \infty$.

2.7 Part (g): Four-dimensional gravity

For four-dimensional gravity, the constant gravitational flux $-4\pi GM$ now must be split over a hypervolume $2\pi^2r^3$ (the volume of a 3-sphere). The exact expression here is not so important, except that this scales as $\propto r^3$. The gravitational force is thus

$$F_r = -\frac{2GmM}{\pi r^3} \tag{27}$$

Then the potential is

$$U(r) = -\frac{2GmM}{3\pi r^2} \tag{28}$$

We see that

$$\alpha = -\frac{2}{3\pi}GmM \tag{29a}$$

$$\beta = -2 \tag{29b}$$

For $\beta = -2$, there are no stable orbits, as we found in part (a). Therefore, four-dimensional gravity actually cannot admit any stable circular orbits at all, around point masses.

2.8 Part (h): Galactic rotation curve

Using the expression for Ω in part (d), we have

$$\Omega \propto r^{\beta/2-1} \tag{30}$$

For $\Omega = v_\phi/r \propto 1/r$, this requires that

$$\beta/2 - 1 = -1 \tag{31}$$

or

$$\beta = 0 \tag{32}$$

Comparing to the gravitational potential $\propto M/r$, we have

$$\text{const.} \propto M(r)/r \tag{33}$$

We see that

$$M(r) \propto r \tag{34}$$

This is incompatible with the expectation that $M(r)$ approach a constant at large r (where most of the visible mass has already been enclosed).

This disagreement between the observed rotation curve of the galaxy and the observed visible mass of the galaxy is one of the classic pieces of evidence for the existence of dark matter.