California Institute of Technology
Physics 2a Recitation
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Act or Diffract
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"It is during our darkest moments that we must focus to see the light." —Aristotle Onassis

Contents

1 Problem Statement

Consider diffraction in two dimensions, where z is oriented along the optical axis and x is perpendicular to it.

Suppose we have a plane wave with wavelength λ (i.e., wavenumber $k = 2\pi/\lambda$) incident upon two tiny apertures placed at $z = 0$, one at $x = -a$ and one at $x = +a$ (i.e., a double slit separated by a distance $2a$). The setup is shown in Figure 1.

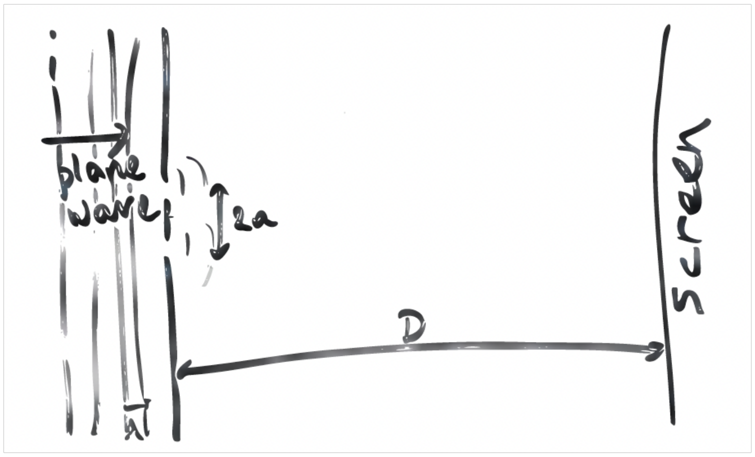

Figure 1: Setup of the problem.

We wish to describe the diffraction pattern that will appear on a screen placed at $z = D$, in the limit where the aperture is far away from the aperture.

(a) The electric field from a point source varies as

$$E = A\frac{e^{ikr}}{r} \tag{1}$$

where r is the distance from the point source and A is an arbitrary normalization. Furthermore, using Huygen's principle, we can treat the apertures as if they are (in-phase, equal-amplitude) point sources.

Write down E for the field on the screen at position x on the screen due to the apertures, in terms of A, k, x, a, and D.

(b) In the limit where $a, x \ll D$ (the **paraxial approximation**), expand the distances "r" which appeared in your answer to part (a), to next-to-leading order.

Where is the x dependence most important, if λ is much smaller than all other length scales?

(c) Keeping only the strongest x dependence in your answer from part (b), approximately where do the peaks in brightness appear on the screen? Troughs?

(d) Now consider a single slit with a finite (but small) width $2b$ centered around $x = 0$. Assume still that each part of the aperture can be modeled as a point source in phase with the rest of aperture, and of uniform amplitude (the **Fraunhofer limit**).

Write an integral yielding E at position x on the screen, using the same approximations as before.

(e) Evaluate the integral in part (d).

2 Problem Solutions

2.1 Part (a): Double slit field

The distance r_+ to the slit at $x = +a$ is

$$r_+ = \sqrt{(x-a)^2 + D^2} \tag{2}$$

Similarly, the distance r_- to the slit at $x = -a$ is

$$r_- = \sqrt{(x+a)^2 + D^2} \tag{3}$$

Then we can write down the field using the superposition principle:

$$E = A\frac{e^{ikr_+}}{r_+} + A\frac{e^{ikr_-}}{r_-} \tag{4}$$

or

$$E = A\frac{e^{ik\sqrt{(x-a)^2+D^2}}}{\sqrt{(x-a)^2+D^2}} + A\frac{e^{ik\sqrt{(x+a)^2+D^2}}}{\sqrt{(x+a)^2+D^2}} \tag{5}$$

2.2 Part (b): Paraxial approximation

We see that r_+ and r_- can be expanded as

$$r_\pm = \sqrt{(x \pm a)^2 + D^2} \approx \sqrt{x^2 \pm 2ax + D^2} \approx r_0\sqrt{1 \pm 2ax/r_0} \approx r_0 \pm ax/r_0 \tag{6}$$

where we have defined $r_0 = \sqrt{x^2 + D^2}$ (the distance to the point on the screen from the midpoint of the two slits).

Then

$$E = A\frac{e^{ik(r_0 - ax/r_0)}}{r_0 - ax/r_0} + A\frac{e^{ik(r_0 + ax/r_0)}}{r_0 + ax/r_0} \tag{7}$$

We see that E depends in this approximation on x in two places:

- In the argument of the complex exponential, due to the phase shift introduced by the different path lengths traversed by the light.

- In the denominator, due to the decrease in amplitude of the electric field as the distance increases.

In the former, the combination kax/r_0 is important when it is comparable to 2π, i.e., we would be able to ignore it if

$$\frac{kax}{r_0} \sim \frac{2\pi ax}{r_0\lambda} \ll 2\pi \tag{8}$$

or

$$ax \ll \lambda r_0 = r_F \tag{9}$$

where the combination λr_0 is called the **Fresnel length scale**. Although $r_0 \gg a, x$, λ is very small, so we cannot generally do this.

However, in the latter, the relevant comparison is

$$\frac{ax}{r_0} \ll r_0 \tag{10}$$

or

$$ax \ll r_0^2 \tag{11}$$

This condition is much easier to satisfy.

Hence, the dependence of x in the argument of the complex exponentials is by far its strongest dependence.

2.3 Part (c): Double slit diffraction pattern

Keeping only the strongest dependence on x in Equation 7, we now have

$$E = \frac{A}{r_0}\left[e^{ik(r_0 - ax/r_0)} + e^{ik(r_0 + ax/r_0)}\right] \tag{12}$$

Peaks occur when the two terms in Equation 12 **constructively interfere**, which is when their phase differences are spaced by some multiple of 2π, i.e.,

$$(kr_0 + kax/r_0) - (kr_0 - kax/r_0) = \frac{4\pi ax}{\lambda r_0} = 2\pi n \tag{13}$$

so that the peaks occur at

$$x = n\frac{r_0\lambda}{2a} = n\frac{r_F^2}{2a} \tag{14}$$

where n is an integer.

Similarly, troughs occur when the two terms are exactly out of phase (so that they cancel out, i.e., **destructively interfere**):

$$\frac{4\pi ax}{\lambda r_0} = 2\pi n + \pi \tag{15}$$

Then

$$x = (n + 1/2)\,\frac{\lambda r_0}{2a} = (n + 1/2)\,\frac{r_F^2}{2a} \tag{16}$$

In other words, the troughs in this limit lie exactly between the peaks.

The diffraction pattern is shown in Figure 2.

Figure 2: Diffraction pattern due to a double slit.

2.4 Part (d): Finite single slit

We now integrate over the slit, again using the principle of superposition:

$$E = \frac{A}{r_0}\int_{-b}^{b} e^{ik(r_0 - xx'/r_0)}\,\mathrm{d}x' = \frac{Ae^{ikr_0}}{r_0}\int_{-b}^{b} e^{-ikxx'/r_0}\,\mathrm{d}x' \tag{17}$$

where we are working in the Fraunhofer limit.

We see that the integral in Equation 17 is a Fourier transform of the aperture.

2.5 Part (e): Finite single slit diffraction pattern

We can evaluate the integral in Equation 17, which is simply that of a (complex) exponential:

$$E = -Aie^{ikr_0}\left(\frac{e^{-ikbx/r_0}}{kx} - \frac{e^{ikbx/r_0}}{kx}\right) \tag{18}$$

We recall that sine is given by

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{19}$$

Then

$$E = -2Ae^{ikr_0}\,\frac{\sin\left(kbx/r_0\right)}{kx} \tag{20}$$

The combination $\sin x/x$ is sometimes called a **sinc function**, i.e.,

$$E = -\frac{2Abe^{ikr_0}}{r_0}\frac{\sin(kbx/r_0)}{kbx/r_0} = -\frac{2Abe^{ikr_0}}{r_0}\text{sinc}\left(\frac{kbx}{r_0}\right) \tag{21}$$

Even though both the numerator and denominator approach 0 as $x \to 0$, the limit of their quotient is 1. A sketch of the intensity of the diffraction pattern (which scales as the square of the sinc function) is shown in Figure 3.

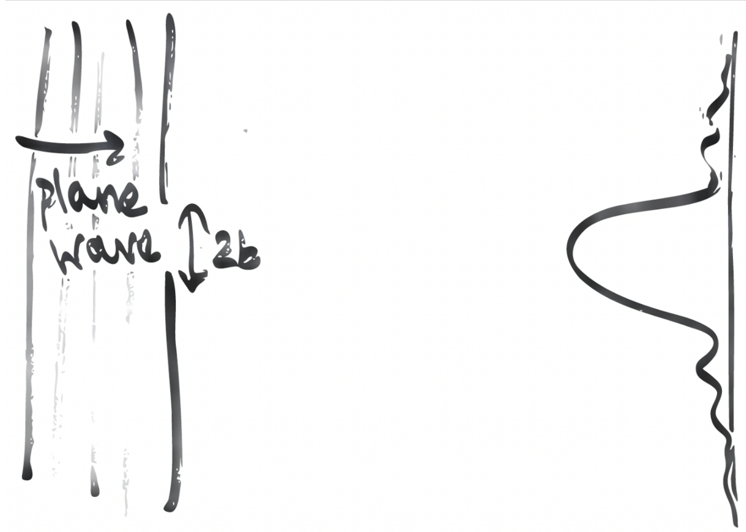

Figure 3: Diffraction pattern due to a finite-width single slit.

Note that the width of the diffraction pattern scales inversely with the width of the slit, i.e., as the slit becomes infinitely narrow, the diffraction pattern becomes infinitely wide. Because the diffraction pattern is just the Fourier transform of the aperture, the diffraction pattern and aperture obey an uncertainty principle.