California Institute of Technology
Physics 2a Recitation
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Do the Diagonal Shuffle
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"I lost my balance when the train pulled away, but a human crumple zone buffered my fall. We stayed like that, half fallen. Diagonal People." —David Mitchell

Contents

1 Eigenproblems

1.1 Solving a set of linear differential equations

Consider the following system of linear differential equations:

$$\begin{aligned} \ddot{x} &= -6x + 2y \\ \ddot{y} &= 2x - 3y \end{aligned} \tag{1}$$

with some initial conditions

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2a}$$

$$\dot{\vec{x}}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2b}$$

Neither of these equations contains only x and y—hence, they cannot be solved individually. We might suspect that we are forced to solve them simultaneously somehow, but it turns out that there is a very clever way to decouple them.

We can rewrite Equations 1 as

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

and, if we define $\vec{x} = (x, y)$ and $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, then we would have

$$\ddot{\vec{x}} = \mathbf{A}\vec{x} \tag{4}$$

If $\mathbf{M}$ were a normal number, then Equation 4 is simply a harmonic oscillator equation. Of course it isn't, and the solution won't just be an oscillation at a single frequency.

However, all is not lost. If it were to turn out that some solutions $\vec{x} = \vec{v}$ obeyed

$$\mathbf{A}\vec{v} = \lambda\vec{v} \tag{5}$$

then, at least for that vector $\vec{v}$, we would be able to replace the *matrix* $\mathbf{A}$ by the *scalar* λ, and then the equation

$$\ddot{\vec{v}} = \mathbf{A}\vec{v} = \lambda\vec{v} \tag{6}$$

would simply have the solution

$$\vec{v} = \vec{C}_1 \cos\left(\sqrt{-\lambda} t\right) + \vec{C}_2 \sin\left(\sqrt{-\lambda} t\right) \tag{7}$$

for some (vectorial) constants $\vec{C}_1$ and $\vec{C}_2$.

Equation 5 is called the **eigenvalue equation** for $\mathbf{A}$, and relates $\mathbf{A}$ to the **eigenvalue** λ and **eigenvector** $\vec{v}$.

We can go about searching for such solutions by rewriting Equation 5 as

$$\mathbf{A}\vec{v} - \lambda\vec{v} = (\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0 \tag{8}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

is the identity matrix. Linear algebra tells us that, if $\vec{v}$ is a nonzero solution to Equation 8, the determinant of $\mathbf{A} - \lambda\mathbf{I}$ must vanish, i.e.,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -6-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = (-6-\lambda)(-3-\lambda) - 4 = 0 \tag{10}$$

Equation 10 is called the **characteristic equation**, and states that some polynomial in λ, called the **characteristic polynomial**, must vanish.

Equation 10 can be simplified into the quadratic

$$\lambda^2 + 9\lambda + 14 = (\lambda + 2)(\lambda + 7) = 0 \tag{11}$$

We therefore see that not all values of λ can be eigenvalues. The only possible eigenvalues λ which could satisfy Equation 5 for some vector $\vec{v}$ are $\lambda = -2$ and $\lambda = -7$. The set of eigenvalues that a matrix (or, more generally, a linear operator) has is called its **spectrum**.

With the allowed values of λ in hand, we can then go back to Equation 8 (without taking the determinant, this time) and row-reducing to find the solution vectors $\vec{v}$.

First, for $\lambda = -2$, we can row reduce $\mathbf{A} + 2\mathbf{I}$:

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \tag{12}$$

We notice that $\mathbf{A} + 2\mathbf{I}$ row-reduces to something with at least one line that is all zeroes. This should always happen, since we have chosen λ so that there would be a space of solutions. The row-reduction in Equation translates to the equation $-2x + y = 0$, i.e.,

$$\vec{v}_{-2} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{13}$$

where x becomes just some constant which we can set as we please (here, we will pick $x = 1$).

We can repeat the process for $\lambda = -7$:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \tag{14}$$

which translates to $x + 2y = 0$, i.e.,

$$\vec{v}_{-7} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{15}$$

We notice that $\vec{v}_{-2}$ and $\vec{v}_{-7}$ are **linearly independent**; they cannot be scaled into each other. Since this is only a two-dimensional space, that means that any vector can be written in terms of the eigenvectors, i.e.,

$$\vec{x} = C_1\vec{v}_{-2} + C_2\vec{v}_{-7} \tag{16}$$

For our initial conditions, for example, we have

$$\vec{x}(0) = \begin{pmatrix}1\\0\end{pmatrix} = C_1\begin{pmatrix}1\\2\end{pmatrix} + C_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}C_1 - 2C_2\\2C_1 + C_2\end{pmatrix} \tag{17}$$

We see that we must choose for $\vec{x}(0)$ the constants $C_1 = 1/5$ and $C_2 = -2C_1 = -2/5$, i.e.,

$$\vec{x}(0) = \frac{1}{5}\vec{v}_{-2} - \frac{2}{5}\vec{v}_{-7} \tag{18}$$

Similarly, for $\dot{x}(0)$, $C_1 = C_2 = 0$.

Recall that our eigenvectors, if they obeyed our differential equation, would have to satisfy

$$\ddot{\vec{v}}_{-2} = -2\vec{v}_{-2} \tag{19a}$$
$$\ddot{\vec{v}}_{-7} = -7\vec{v}_{-7} \tag{19b}$$
$$\tag{19c}$$

which are simple harmonic oscillator equations, i.e., we know they must be solved by

$$\vec{v}_{-2}(t) = \vec{v}_{-2}(0)\cos\left(\sqrt{2}t\right) + \frac{1}{\sqrt{2}}\dot{\vec{v}}_{-2}(0)\sin\left(\sqrt{2}t\right) \tag{20a}$$
$$\vec{v}_{-7} = \vec{v}_{-7}(0)\cos\left(\sqrt{7}t\right) + \frac{1}{\sqrt{7}}\dot{\vec{v}}_{-7}\sin\left(\sqrt{7}t\right) \tag{20b}$$
$$\tag{20c}$$

Since $\mathbf{M}$ is linear, we know from the decomposition in Equation 18 that

$$\vec{x}(t) = \frac{1}{5}\vec{v}_{-2}\cos\left(\sqrt{2}t\right) - \frac{2}{5}\vec{v}_{-7}\cos\left(\sqrt{7}t\right) \tag{21}$$

where we have set the prefactors to match the boundary conditions. This is the solution for $\vec{x}(t)$ at all times, and was achieved in a nice closed form even though $\vec{x}(0)$ was not actually itself an eigenvector.

Written out, Equation 21 becomes

$$\vec{x}(t) = \begin{pmatrix}\frac{1}{5}\cos\left(\sqrt{2}t\right) + \frac{4}{5}\cos\left(\sqrt{7}t\right)\\ \frac{2}{5}\cos\left(\sqrt{2}t\right) - \frac{2}{5}\cos\left(\sqrt{7}t\right)\end{pmatrix} \tag{22}$$

or, taken back apart,

$$x(t) = \frac{1}{5}\cos\left(\sqrt{2}t\right) + \frac{4}{5}\cos\left(\sqrt{7}t\right) \tag{23a}$$
$$y(t) = \frac{2}{5}\cos\left(\sqrt{2}t\right) - \frac{2}{5}\cos\left(\sqrt{7}t\right) \tag{23b}$$

We thus see the root of the power of eigenvectors: they allow us to replace operators with numbers, and we *understand* numbers.

1.2 Simple harmonic oscillation as an eigenvalue problem

The general simple harmonic oscillator equation is

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{24}$$

In Section 1.1, we made extensive use of the fact that we know the answer to Equation 24 is oscillatory.

However, Equation 24 is itself an eigenvalue equation. It can be rearranged to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x = -\omega^2 x \tag{25}$$

where we now see that Equation 25 requires that the solution x be an **eigenfunction** of a linear operator $\mathrm{d}^2/\mathrm{d}t^2$.

We like when things solve Equation 24 because it means that we can replace the second derivative operator (whose behavior is complicated) by a constant eigenvalue $-\omega^2$, with which we can do basic algebra.

The eigenfunctions of $\mathrm{d}^2/\mathrm{d}t^2$ are sines and cosines (alternatively, exponentials with possibly complex exponents). We like working with sines and cosines not because every solution will take their form, but because we hope (with retrospective justification) that any other possible solutions can be decomposed into them.

2 Problem Statement

In this Problem, we will investigate the behavior of N "small" springs (with spring constants k_i) attached to a little "big" spring (with spring constant q). The setup is shown in Figure 1.

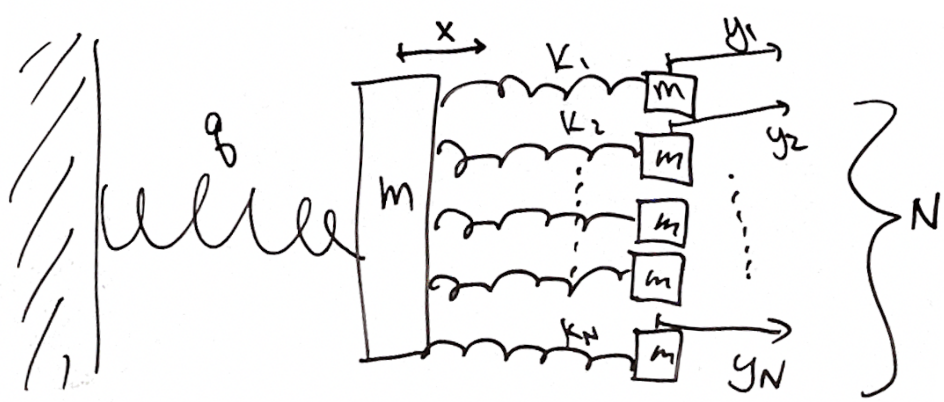

Figure 1: Setup of the problem.

Define the position of the mass on the big spring with respect to equilibrium as x, and the absolute position of the masses on the small springs as y_i (where $i = 1, 2, \ldots, N$).

Assume all of the masses m are the same.

(a) Write down Newton's second law for each of the small springs.

Call each of the oscillator's frequencies ω_i.

(b) Write down Newton's second law for the big spring.

Call the big spring's oscillation frequency (with respect to a single mass m) ν.

(c) Combine your answers (a) and (b) into a matrix equation relating $\ddot{\vec{x}}$ and $\vec{x}$.

(d) Consider the $N = 1$ case.

If the square frequencies are spaced evenly, find the eigenvalues of the matrix in part (d) in the limit where the spacing is small.

(e) Use some numerical software to solve for the eigenvalues of the matrix numerically in part (d).

Assume that $\omega_1^2 - \nu^2 = \omega_2 - \omega_1^2 = \omega_2^3 - \omega_3^3 = \ldots$ is a small (i.e., the oscillators have very similar frequencies, with the lowest being that of the big spring).

Describe qualitatively hat happens to the spectra.

3 Problem Solutions

3.1 Part (a): Small springs

For each small spring, we can write down Newton's second law, $F = ma$. The ith mass only feels the force of the spring that it is attached to:

$$F = -k_i(y_i - x) \tag{26}$$

The acceleration is simply

$$a = \ddot{y}_i \tag{27}$$

and our equation of motion for the ith mass becomes

$$m\ddot{y}_i = -k_i(y_i - x) \tag{28}$$

or

$$\ddot{y}_i = \omega_i^2 x - \omega_i^2 y_i \tag{29}$$

where

$$\omega_i^2 \equiv \frac{k_i}{m} \tag{30}$$

There are N equations of the form shown Equations 29, one for each value of i.

3.2 Part (b): Big spring

The mass attached to the big spring feels a force from the big spring as well as every small spring:

$$F = -qx - \sum_j k_j(x - y_j) \tag{31}$$

The acceleration is

$$a = \ddot{x} \tag{32}$$

Then

$$m\ddot{x} = -qx - \sum_j k_j(x - y_j) \tag{33}$$

or

$$\ddot{x} = -\chi^2 x + \sum_j \omega_j^2 y_j \tag{34}$$

where we have defined

$$\chi^2 \equiv \nu^2 + \sum_j \omega_j^2 \tag{35}$$

where

$$\nu^2 \equiv \frac{q}{m} \tag{36}$$

3.3 Part (c): Matrix equation

We can compile Equation 34 and Equations 29 into a system of $N + 1$ linear differential equations:

$$\ddot{x} = -\chi^2 x + \omega_1^2 y_1 + \omega_2^2 y_2 + \cdots + \omega_N^2 y_N^2 \tag{37a}$$
$$\ddot{y}_1 = \omega_1^2 x - \omega_1^2 y_1 \tag{37b}$$
$$\ddot{y}_2 = \omega_2^2 x - \omega_2^2 y_2 \tag{37c}$$
$$\vdots \tag{37d}$$
$$\ddot{y}_N = \omega_N^2 x - \omega_N^2 y_N \tag{37e}$$
$$\tag{37f}$$

Equations 37 can be written as a matrix equation:

$$\ddot{x} = -\mathbf{\Omega}\vec{x} \tag{38}$$

where

$$\vec{x} = \begin{pmatrix} x \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \tag{39}$$

and

$$\mathbf{\Omega} = \begin{pmatrix} \chi^2 & -\omega_1^2 & -\omega_2^2 & \cdots & -\omega_N^2 \\ -\omega_1^2 & \omega_1^2 & 0 & \cdots & 0 \\ -\omega_2^2 & 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\omega_N^2 & 0 & 0 & \cdots & \omega_N^2 \end{pmatrix} \tag{40}$$

3.4 Part (d): $N = 1$ case

When $N = 1$,

$$\mathbf{\Omega} = \begin{pmatrix} \chi^2 & -\omega_1^2 \\ -\omega_1^2 & \omega_1^2 \end{pmatrix} = \nu^2 \begin{pmatrix} 2+\delta & -1-\delta \\ -1-\delta & 1+\delta \end{pmatrix} \tag{41}$$

where $\delta = (\omega_1^2 - \nu^2)/\nu^2$.

We can diagonalize Equation 41 by solving the characteristic polynomial:

$$\det\bigl(\mathbf{\Omega} - \Omega^2\mathbf{I}\bigr) = 0 \tag{42}$$

where $\mathbf{I}$ is the identity matrix and Ω^2 is the eigenvalue. Note that the purpose of finding the eigenvalues is that, for an eigenvector, $\mathbf{\Omega}$ (a complicated matrix) can be replaced by a scalar Ω^2 (which is easier to think about).

In this case, we have

$$\begin{aligned} \det\bigl(\mathbf{\Omega} - \Omega^2\mathbf{I}\bigr) &= \begin{vmatrix} (2+\delta)\nu^2 - \Omega^2 & (-1-\delta)\nu^2 \\ (-1-\delta)\nu^2 & (1+\delta)\nu^2 - \Omega^2 \end{vmatrix} \\ &= ((2+\delta)\nu^2 - \Omega^2)((1+\delta)\nu^2 - \Omega^2) - (-1-\delta)^2\nu^4 \\ &= \Omega^4 - (3+2\delta)\nu^2\Omega^2 + (1+\delta)\nu^4 \end{aligned} \tag{43}$$

where we have dropped all terms $\mathcal{O}(\delta^2)$.

This quadratic (in Ω^2) is solved by

$$\begin{aligned} \Omega_\pm^2 &= \frac{(3+2\delta) \pm \sqrt{(3+2\delta)^2 - 4(1+\delta)}}{2} \\ &= \frac{(3+2\delta) \pm \sqrt{5+8\delta}}{2} \\ &= \frac{(3+2\delta) \pm (\sqrt{5} + 4\delta/\sqrt{5})}{2} \end{aligned} \tag{44}$$

or

$$\Omega_\pm^2 = \left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right) + \left(1 \pm \frac{2}{\sqrt{5}}\right)\delta \tag{45}$$

3.5 Part (e): Large N solution

We can numerically solve the characteristic equation $\det\left(\mathbf{\Omega} - \Omega^2\mathbf{I}\right) = 0$ for a range of N (in this case, using the SCIPY python package). The eigenvalues are shown in Figure 2.

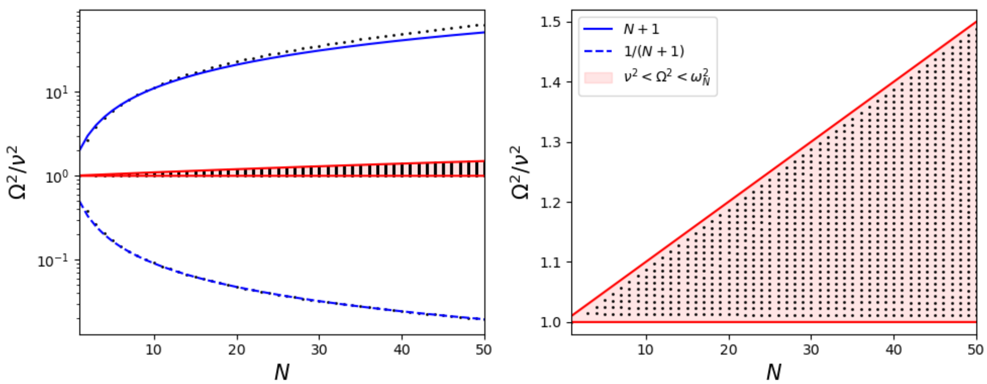

Figure 2: Eigenvalues of $\mathbf{\Omega}$, for $\delta = 10^{-3}$.

Plotting these eigenfunctions reveals that almost all (except two) eigenvalues are packed within a band $\nu^2 < \Omega^2 < \omega_N^2$, between the squared oscillation frequencies of the individual springs.

However, two eigenvalues are very far from this region. A large eigenvalue appears around $\Omega^2 = (N+1)\nu^2$, and a small eigenvalue appears around $\Omega^2 = \nu^2/(N+1)$.