California Institute of Technology
Physics 2a Recitation
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Getting Tense
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"Tension is the cornerstone of any good story." —*Eric Nylund*

Contents

1 Problem Statement

For a string with constant tension T and constant mass density μ, a small perturbation $f = f(x,t)$ will follow the wave equation:

$$\mu \frac{\partial^2 f}{\partial t^2} = T \frac{\partial^2 f}{\partial x^2} \tag{1}$$

This can be rewritten as

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{2}$$

where $c^2 = T/\mu$ is the squared wave speed.

However, when T and μ can be functions of position, i.e., $T = T(x)$ and $\mu = \mu(x)$, it can be shown that a perturbation will now follow

$$\mu \frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial x}\left(T \frac{\partial f}{\partial x} \right) = T \frac{\partial^2 f}{\partial x^2} + \frac{\partial T}{\partial x}\frac{\partial f}{\partial x} \tag{3}$$

Equation 3 looks very similar to the wave equation, but contains an extra term $\propto \partial T/\partial x$.

In the following Problem, we will investigate the impact of this term, and when it can be ignored.

(a) Divide Equation 3 by Tf' and show that the first term on the right-hand side dominates when the relative spatial variation of f' is more rapid than T.

(b) In a normal mode (i.e., standing wave), we can replace

$$\frac{\partial^2}{\partial t^2} \to \omega^2 \tag{4}$$

for real ω, so that Equation 3 becomes

$$\mu\omega^2 f = Tf'' + T'f' \tag{5}$$

Consider a wave in the string which takes the form

$$f(x) = A(x) e^{i \int^x k(x')\,dx'} \tag{6}$$

where we will assume that k is very large compared to A' and k'/k. This assumption, called the **Wentzel–Kramers–Brillouin (WKB) approximation**, basically states that it makes to talk about the wave as having a wavelength *at a given point*. If, in contrast, $A(x)$ or $k(x)$ varied comparably quickly (spatially) as $\int^x k(x')\,dx'$, there would be no concrete way to tell apart a change in amplitude or wavelength from the wavelike oscillation.

Compute f' and f'' in this limit.

(c) Plug in your expressions from part (b) into Equation 3 and keep only the largest terms. Do not yet assume anything about how quickly the tension varies in space.

Derive an equation which can be solved to yield k.

(d) Solve for k in your expression from part (c).

Find k in the limiting cases where $2\omega\sqrt{\mu/T} \quad |T'/T|$ and $2\omega\sqrt{\mu/T} \quad |T'/T|$. Comment on your findings.

Consider now a rope with (possibly non-constant) μ with a mass m attached to the bottom. Call y the vertical distance measured from the bottom of the rope.

Figure 1 shows the setup described above.

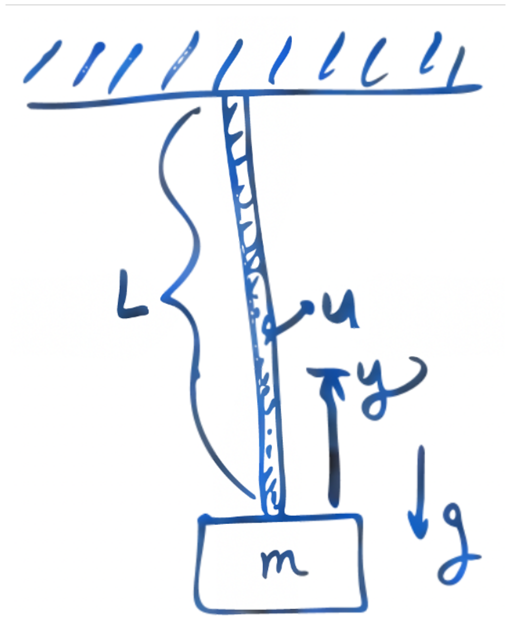

Figure 1: Setup of the problem.

(e) The tension in the rope at a height y must balance the force of gravity needed to hold up the rope beneath that height.

Use this information to write an expression for T in terms of μ.

(f) The wave speed (under the WKB approximation) is

$$c^2 = \frac{T}{\mu} \tag{7}$$

Find how the mass density μ must depend on y to keep c^2 constant. In other words, even though T varies with height in a way determined by μ, find the function μ that somehow keeps the wave speed in Equation 7 anyway.

(g) Would waves on a rope as described in part (f) behave exactly the same as a constant tension, constant mass density rope?

(h) Now consider a uniform rope with mass density μ hanging from the ceiling without any mass attached, so that

$$T(y) = \mu g y \tag{8}$$

The rope is a length L, and thus the ceiling imposes a boundary condition on the wave: $f(x = L) = 0$. The bottom end is free.

Solve for the modes of Equation 3 *exactly*. What values of ω are allowed?

2 Problem Solutions

2.1 Part (a): Ignoring the spatial dependence of tension

When dividing through Equation 3 by Tf', we have

$$\frac{\mu \ddot{f}}{T f'} = \frac{f''}{f'} + \frac{T'}{T} \tag{9}$$

We see that the first term is the relative change of f' with x, and the second term is the relative change of T with x. In other words, to ignore the second term, it is required that f' varies much more quickly than T, in a relative sense.

To be concrete, this term can be ignored so long as either f (and thus f') varies very quickly in space, or T varies very slowly.

2.2 Part (b): Calculating spatial derivatives

The first and second spatial derivatives of f are

$$f' = A' e^{i \int^x k(x')\,dx'} + ikAe^{i \int^x k(x')\,dx'} = \left(\frac{A'}{A} + ik\right) f \tag{10a}$$

$$f'' = (A'' + ik'A + ikA')e^{i \int^x k(x')\,dx'} + ik(A' + ikA)e^{i \int^x k(x')\,dx'} = \left(\frac{A''}{A} + ik' + 2ik\frac{A'}{A} - k^2\right) f \tag{10b}$$

Keeping only the largest terms (those highest-order in k), we have

$$f' \approx ikf \tag{11a}$$

$$f'' \approx -k^2 f \tag{11b}$$

We see that the WKB approximation essentially amounts to assuming that $f \sim e^{ikx}$, but allowing k (and thus the wavelength) to very *slowly* with position.

2.3 Part (c): Expression for k

Plugging Equations 11 into Equation 5, we have

$$\mu\omega^2 f = -k^2 T f + ikT' f \tag{12}$$

Each term has a factor of f, which can be divided out. We are then left with the quadratic

$$k^2 - ik\frac{T'}{T} - \frac{\mu\omega^2}{T} = 0 \tag{13}$$

2.4 Part (d): Solving for k

Using the quadratic formula, Equation 13 becomes

$$k = \frac{i(T'/T) \pm \sqrt{(T'/T)^2 + 4\mu\omega^2/T}}{2} = \pm\sqrt{\frac{\mu}{T}}\omega\sqrt{1 - \frac{T'^2}{4T\mu\omega^2}} + \frac{i}{2}\left(\frac{T'}{T}\right) \tag{14}$$

First, consider the case where $2\omega\sqrt{\mu/T} \gg |T'/T|$ (i.e., the tension varies very slowly compared to the length scale defined by the frequency).

Then Equation 15 becomes

$$k \approx \pm\sqrt{\frac{\mu}{T}}\omega \tag{15}$$

We see that Equation 15 gives a *real* k (i.e., an oscillatory, or **propagating**, wave), which follows the dispersion relation of a string with constant tension:

$$\frac{\omega^2}{k^2} = \frac{T}{\mu} \tag{16}$$

This demonstrates that, for high frequency waves, a wave behaves very similarly to a wave on a constant-tension, constant-mass density string.

We next consider the case where $2\omega\sqrt{\mu/T} \ll |T'/T|$, i.e., the tension varies (relatively) very quickly compared to some length scale defined by ω. Then we see that

$$k \approx \pm\frac{i}{2}\left(\frac{T'}{T}\right) + \frac{i}{2}\left(\frac{T'}{T}\right) = \text{either } i\left(\frac{T'}{T}\right) \text{ or } 0 \tag{17}$$

The first of these wavenumbers is imaginary, which means that the wave is **evanescent**, i.e., it either grows or decays with x. It can be checked that the sign of k implies that the wave will decay in the direction where T'/T increases.

The other solution, $k \approx 0$, seems to violate the hypothesis that k is very large. This seems to imply that the WKB approximation is breaking down.

Another way of seeing the evanescent behavior of the wave in this limit is by ignoring the *first* term on the right hand side of Equation 3, and noticing that the equation then applies a spatial exponential decay.

Figure 2 summarizes this behavior.

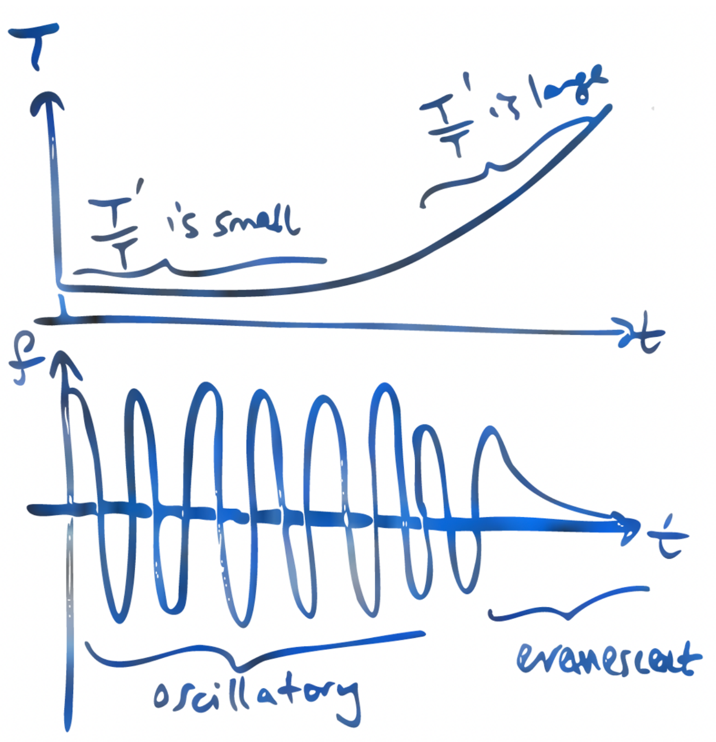

Figure 2: A sketch of the spatial structure of a mode in a non-uniformly tense string.

2.5 Part (e): Tension in a hanging string

We can equate the tension $T = T(y)$ with the gravitational force needed to support the rope underneath that point:

$$T(y) = g \int_0^y \mu(y') \, \mathrm{d}y' + mg \tag{18}$$

2.6 Part (f): Mass density of a rope with a constant wave speed

The (squared) wave speed is

$$c^2 = \frac{T}{\mu} = \frac{g \int_0^y \mu(y') \, \mathrm{d}y' + mg}{\mu(y)} \tag{19}$$

We can rearrange this as

$$c^2 \mu(y) = g \int_0^y \mu(y') \, \mathrm{d}y' + mg \tag{20}$$

We can then take a derivative with y to obtain

$$c^2 \frac{\mathrm{d}\mu}{\mathrm{d}y} = \mu g \tag{21}$$

We can separate variables to write the integral

$$\int_{\mu(0)}^{\mu(y)} \frac{d\mu'}{\mu'} = \frac{g}{c^2}\int_0^y dy' \tag{22}$$

This yields

$$\ln\left(\frac{\mu(y)}{\mu(0)}\right) = \frac{gy}{c^2} \tag{23}$$

This gives

$$\mu(y) = \mu(0)e^{gy/c^2} \tag{24}$$

Note that $\mu(0)$ is just some constant, and c is just the constant wave speed whose value we are not particularly concerned with. Thus, we see that the mass density would have to increase exponentially with height in order to keep c constant.

Note that, when $gy \ll c^2$ always (i.e., the hanging mass dominates the rope tension), we are reduced to the case of a uniform-density rope.

2.7 Part (g): Comparison with uniform-density rope

No; part (f) gives the mass density required to keep the combination of variables T/μ constant, but this is only the wave speed for small wavelength oscillations (i.e., those which obey the WKB approximation).

In general, when the wavelength is large, waves may reflect or become evanescent due to the second term in Equation 3.

2.8 Part (h): Exact mode solution for hanging rope

For the tension defined in Equation 8, Equation 3 becomes

$$\mu\omega^2 f(y) = \mu g y f''(y) + \mu g f'(y) \tag{25}$$

where we have taken $\partial^2/\partial t^2 \to -\omega^2$.

This can be rewritten as

$$y f''(y) + f'(y) + \frac{\omega^2}{g} f(y) = 0 \tag{26}$$

Equation 26 (which does not depend on μ) can be transformed to the **Bessel equation**, and is solved by Bessel functions:

$$f(x) = AJ_0\left(2\sqrt{\frac{\omega^2 y}{g}}\right) + BY_0\left(2\sqrt{\frac{\omega^2 y}{g}}\right) \tag{27}$$

where J_0 is the zeroth-order Bessel function of the first kind, and Y_0 is the zeroth-order Bessel function of the second kind.

Because $Y_0(x)$ diverges when $y = 0$, we discard it for unphysicality. Then we are left with

$$f(x) = AJ_0\left(2\sqrt{\frac{\omega^2 y}{g}}\right) \tag{28}$$

We can then impose the boundary condition at $y = L$:

$$f(L) = AJ_0\left(2\sqrt{\frac{\omega^2 L}{g}}\right) = 0 \tag{29}$$

This is only possible when

$$2\sqrt{\omega^2 L/g} = \beta_n \tag{30}$$

where β_n is the nth root of J_0. Note that, unlike in the case of imposing boundary conditions on a sinusoidal wave (where the roots are equally spaced by a multiple of π), the roots β_n are not equally spaced. However, they still impose a quantization condition.

In particular, the only allowed frequencies are

$$\omega_n = \frac{1}{2}\beta_n\sqrt{\frac{g}{L}} \tag{31}$$

Then the general mode solution becomes

$$f_n(x) = AJ_0\left(\beta_n\sqrt{\frac{y}{L}}\right) \tag{32}$$

The Bessel function J_0 and the modes $f_n(x)$ are shown in Figure 3.