California Institute of Technology
Physics 2a Recitation
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String Theory
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"You will find truth more quickly through delight than gravity. Let out a little more string on your kite." —Alan Cohen

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1 Problem Statement

Consider a series of (small) masses m connected together by a string with tension T. The masses are spaced apart by a (small) distance δx, and are constrained to only move up and down. The (small) vertical displacement of the nth mass is y_n. The setup is shown in Figure 1.

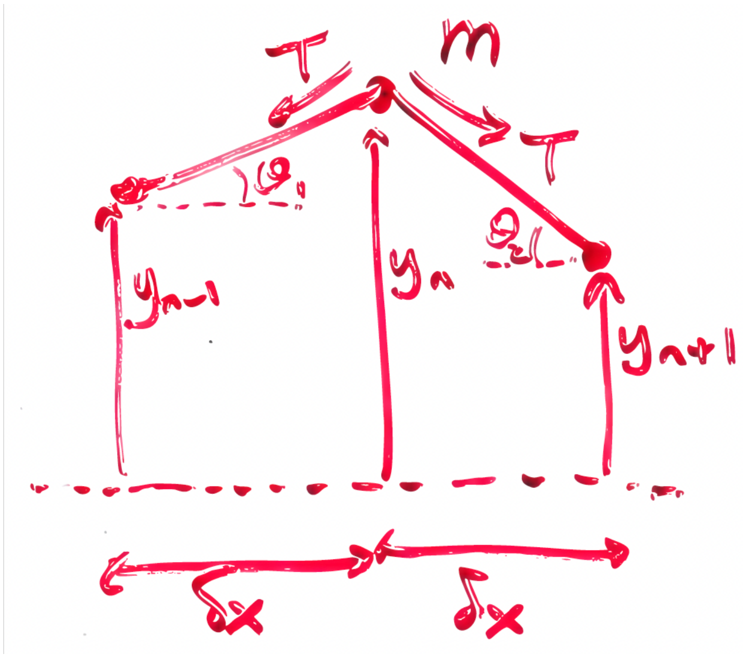

Figure 1: Setup of the problem.

(a) Using Newton's second law, write an equation for $\ddot{y}_n$.

(b) Define the linear mass density as $\mu = m/\delta x$ (the mass per unit length of the string).

Show that your answer to part (a) becomes a wave equation, when all small quantities are taken to be infinitesimal.

From now on, write your answers with respect to μ and c, rather than T.

(c) Derive the kinetic and potential energy (length) density in terms of $\ddot{y}$ and y''.

(d) Suppose the string is fixed at $x = 0$ and $x = L$ (i.e., $y(0) = y(L) = 0$). Show that the only allowed values of the wavenumber $k = \omega/c$ are

$$k = \frac{n\pi}{L} \tag{1}$$

where $n = 1, 2, 3, \ldots$.

(e) Suppose the string is plucked in the middle as a triangle wave:

$$y(t=0) = \begin{cases} 2Ax/L & x < L/2 \\ (2A/L)(L-x) & x > L/2 \end{cases} \tag{2}$$

If we would like to decompose y into sinusoidal signals, i.e.,

$$y(t=0) = A_1 \sin\left(\frac{\pi x}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right) + A_3 \sin\left(\frac{3\pi x}{L}\right) + \ldots + A_n \sin\left(\frac{n\pi x}{L}\right) + \ldots \quad (3)$$

What are the coefficients A_n?

2 Problem Solutions

2.1 Part (a): Difference equation

The nth mass (at a height y_n) is connected by the string to its neighboring masses at heights y_{n-1} and y_{n+1}. The angles from the horizontal involved (labeled in Figure 1) are

$$\theta_1 \approx \tan\theta_1 = \frac{y_n - y_{n-1}}{\delta x} \quad (4a)$$

$$\theta_2 \approx \tan\theta_2 = \frac{y_{n+1} - y_n}{\delta x} \quad (4b)$$

Then the net force acting on the nth mass is

$$F = m\ddot{y}_n = -T\sin\theta_1 - T\sin\theta_2 \approx -T(\theta_1 + \theta_2) \approx T\frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x} \quad (5)$$

so that the desired difference equation is

$$m\ddot{y}_n = T\frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x} \quad (6)$$

2.2 Part (b): Differential equation

We notice that the position derivatives of y are approximated by the differences

$$\frac{y_n - y_{n-1}}{\delta x} \approx y'(x - \delta x/2) \quad (7a)$$

$$\frac{y_{n+1} - y_n}{\delta x} \approx y'(x + \delta x/2) \quad (7b)$$

Then the second derivative is approximated by

$$y''(x) \approx \frac{y'(x + \delta x/2) - y'(x - \delta x/2)}{\delta x} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x^2} \quad (8)$$

We can then divide Equation 6 by δx to obtain

$$\frac{m}{\delta x}\ddot{y} = \mu\ddot{y} = T\frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x^2} = Ty'' \quad (9)$$

and thus we obtain the wave equation

$$\ddot{y} = c^2 y'' \quad (10)$$

where the wave speed is

$$c = \sqrt{\frac{T}{\mu}} \quad (11)$$

2.3 Part (c): Energy density

The kinetic energy K of a mass m is simply

$$K = \frac{1}{2} m \dot{y}^2 \tag{12}$$

Therefore, the linear density k of kinetic energy is

$$k = \frac{1}{2} \mu \dot{y}^2 \tag{13}$$

The force acting on the mass is simply the force shown in Equation 5:

$$F \approx T \frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x} \approx T y'' \delta x \tag{14}$$

To get the potential energy of the particle, we should integrate the force F back to the equilibrium point $y = 0$ (from a position $y = \delta y$):

$$\begin{aligned} U &= -\int_{\delta y}^{0} F \, \mathrm{d}y \\ &= T\delta x \int_0^{\delta y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) \mathrm{d}y \\ &= T\delta x \int_0^{\delta y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) \frac{\partial y}{\partial t} \, \mathrm{d}t \\ &= \frac{1}{2} T\delta x \int_0^{\delta y} \frac{\partial}{\partial t} \left(\frac{\partial (y')^2}{\partial t} \right) \mathrm{d}t \\ &= \frac{1}{2} T\delta x \int_0^{\delta t} \frac{\partial y'^2}{\partial t} \, \mathrm{d}t \\ &= \frac{1}{2} T\delta x (y'^2(\delta t) - y'^2(0)) \\ &= \frac{1}{2} T\delta x \left(\frac{\partial y}{\partial x} \right)^2 \end{aligned} \tag{15}$$

where we ignore the constant term. Dividing through by δx, the potential energy density is

$$u = \frac{1}{2} T y'^2 = \frac{1}{2} \mu c^2 y'^2 \tag{16}$$

The total energy density $\mathcal{E}$ is then

$$\mathcal{E} = k + u = \frac{1}{2} \mu \dot{y}^2 + \frac{1}{2} \mu c^2 y'^2 = \frac{1}{2} \mu (\dot{y}^2 + c^2 y'^2) \tag{17}$$

2.4 Part (d): Quantization

A sinusoidal signal (at $t = 0$) has

$$y(x) = A\sin(kx) + B\cos(kx) \tag{18}$$

Fixing $y(x = 0) = 0$ gives

$$y(0) = B = 0 \tag{19}$$

so that $B = 0$. Then fixing $y(x = L) = 0$ gives

$$A\sin(kx) = 0 \tag{20}$$

If A is not zero (which is required for the solution to be nontrivial), this means that kx must be an integer multiple of π, i.e.,

$$k = \frac{n\pi}{L} \tag{21}$$

where n is an integer.

Notice that $n = 0$ just produces a zero signal, and any negative values of n are equivalent to the positive values (by flipping the sign of A).

2.5 Part (e): Triangle wave

Let q be an integer. We see that, if $q \neq 0$, we can write the integral

$$\int_0^L \cos\left(\frac{q\pi x}{L}\right)\,\mathrm{d}x = \frac{L}{q\pi}\left(\sin\left(q\pi\right) - \sin\left(0\right)\right]_0^L = 0 \tag{22}$$

On the other hand, if $q = 0$, then this integral becomes

$$\int_0^L \cos\left(\frac{q\pi x}{L}\right)\,\mathrm{d}x = \int_0^L \mathrm{d}x = L \tag{23}$$

We then consider the integral

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \tag{24}$$

for integers n and m. We note the trigonometric identities

$$\cos\left((n-m)\frac{\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \tag{25a}$$

$$\cos\left((n+m)\frac{\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) - \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \tag{25b}$$

We can add Equations 25 to obtain

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left[\cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m)\frac{\pi x}{L}\right)\right] \tag{26}$$

Then the integral in Equation 24 becomes

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = \frac{1}{2}\left[\int_0^L \cos\left((n-m)\frac{\pi x}{L}\right)\,\mathrm{d}x - \int_0^L \cos\left((n+m)\frac{\pi x}{L}\right)\,\mathrm{d}x\right] \tag{27}$$

For $n, m \geq 1$, $n+m$ cannot be zero, so the second term has to integrate to zero (by Equation 22). The first term vanishes when $n \neq m$, but becomes L when $n = m$ (by Equation 23). Then

$$\frac{1}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \tag{28}$$

We thus see that integrating a function against $L^{-1}\sin(n\pi x/L)$ has the effect of pulling out the coefficient A_n.

We can then write

$$\begin{aligned} A_n &= \frac{1}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) y(t=0)\,\mathrm{d}x \\ &= \frac{2A}{L^2}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x + \frac{2A}{L}\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x - \frac{2A}{L^2}\int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{2A}{n^2\pi^2}\int_0^{n\pi/2} u\sin u\,\mathrm{d}u + \frac{2A}{n\pi}\int_{n\pi/2}^{n\pi}\sin u\,\mathrm{d}u - \frac{2A}{n^2\pi^2}\int_{n\pi/2}^{n\pi} u\sin u\,\mathrm{d}u \end{aligned} \tag{29}$$

Now, using

$$\int x\sin x\,\mathrm{d}x = \sin x - x\cos x \tag{30}$$

these sub-integrals become

$$\frac{2A}{n^2\pi^2}\int_0^{n\pi/2} u\sin u\,\mathrm{d}u = \frac{2A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{A}{n\pi}\cos\left(\frac{n\pi}{2}\right) \tag{31a}$$

$$\frac{2A}{n\pi}\int_{n\pi/2}^{n\pi}\sin u\,\mathrm{d}u = -\frac{2A}{n\pi}\cos(n\pi) + \frac{2A}{n\pi}\cos\left(\frac{n\pi}{2}\right) \tag{31b}$$

$$-\frac{2A}{n^2\pi^2}\int_{n\pi/2}^{n\pi} u\sin u\,\mathrm{d}u = -\frac{2A}{n^2\pi^2}\sin(n\pi) + \frac{2A}{n\pi}\cos(n\pi) + \frac{2A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{A}{n\pi}\cos\left(\frac{n\pi}{2}\right) \tag{31c}$$

Then

$$A_n = \frac{4A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2A}{n^2\pi^2}\sin(n\pi) \tag{32}$$

This coefficient behaves differently for even and odd coefficients. We see that

$$A_n = \begin{cases} 0 & n\,\text{even} \\ \frac{4A}{n^2\pi^2}(-1)^{(n-1)/2} & n\,\text{odd} \end{cases} \tag{33}$$

We see in Figure 2 that a triangle "plucked" wave initial condition can be decomposed into harmonics of the string, with the inclusion of higher-n harmonics successively improving the approximation. The wave does not remain a triangle shape for long, since each component of the wave oscillates with a different frequency.

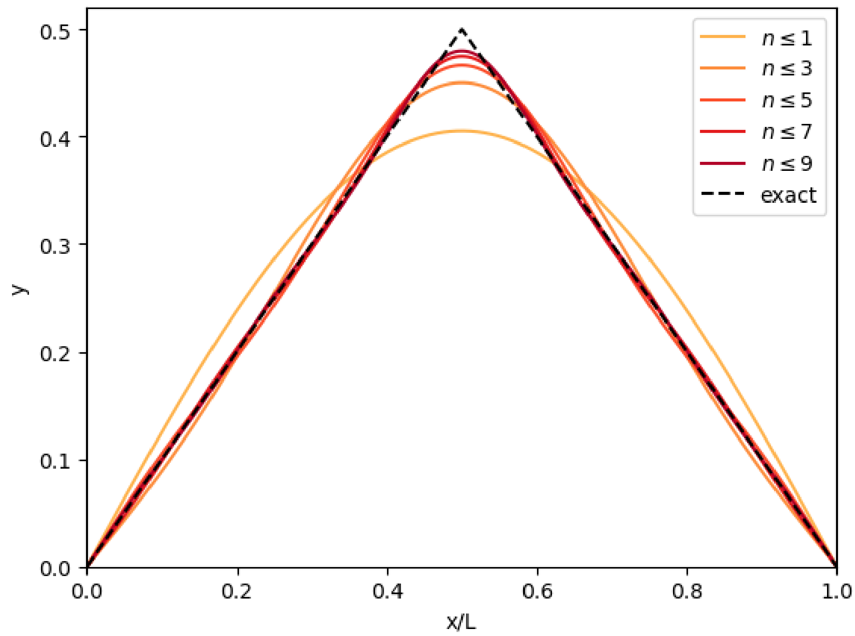

Figure 2: Approximations to a triangle wave using sine waves.