California Institute of Technology
Physics 2a Recitation
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The Ultimate Spring Problem
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"Success comes in waves." —Guy Pearce

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1 Problem Statement

Consider many masses m which are all connected in sequence by springs with spring constants k and lengths δx (Figure 1).

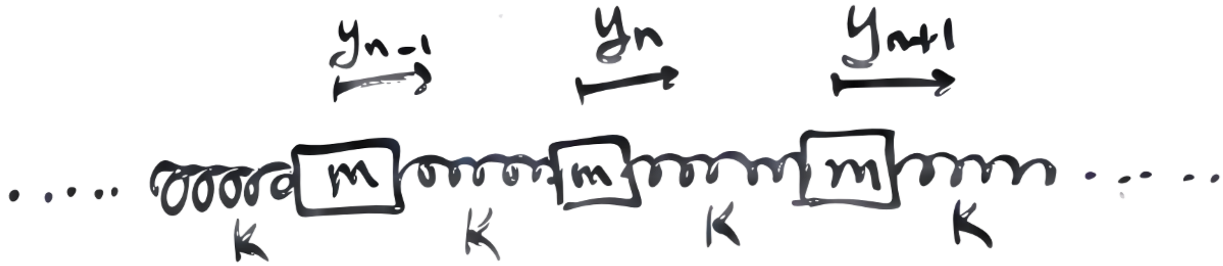

Figure 1: Setup of the problem.

Label the position of the nth mass with respect to the equilibrium position as y_n. Remember that the nth mass is attached to the $(n-1)$th and $(n+1)$th masses.

(a) Use Newton's second law to write a difference equation relating y_n to y_{n-1} and y_{n+1}.

(b) In the limit where the displacements and δx is small, show that your answer to part (a) becomes a wave equation in $y(x,t)$.

What is the wave speed?

(c) Consider a separable solution $y(x,t) = X(x)T(t)$.

Show that your answer to part (b) can be rewritten an equation with two terms, one only depending on x and the other only depending on t.

(d) Show that $X(x)$ and $T(t)$ are both solved by oscillatory functions with "frequencies" k and ω, respectively.

How are k and ω related?

(e) Suppose that the chain of springs is constrained at $x = 0$ and $x = L$ to not move (e.g., it is pinned to the wall, $y(0,t) = y(L,t) = 0$).

What are the allowed values of ω and k?

2 Problem Solutions

2.1 Part (a): Difference equation

We can write down Newton's second law, $F = ma$,

$$m\ddot{y}_n = -k(y_n - y_{n-1}) - k(y_n - y_{n+1}) \tag{1}$$

2.2 Part (b): Differential equation

We notice that, to close approximation,

$$y'_n = \frac{y_n - y_{n-1}}{\delta x} \tag{2a}$$
$$y'_{n+1} = \frac{y_{n+1} - y_n}{\delta x} \tag{2b}$$
$$\tag{2c}$$

where primes denote spatial derivatives (not with respect to y_n but with respect to x, the spacing between the masses in equilibrium). Note that y'_n should be viewed as being evaluated between y_{n-1} and y_n, not at any specific lattice point.

Then Equation 1 becomes

$$m\ddot{y}_n = k\delta x(y'_{n+1} - y'_n) \tag{3}$$

Note also that

$$y''_n = \frac{y'_{n+1} - y'_n}{\delta x} \tag{4}$$

so that

$$m\ddot{y}_n = k\delta x^2 y''_n \tag{5}$$

We note that

$$\lambda = \frac{m}{\delta x} = \text{linear mass density} \tag{6a}$$
$$\kappa = k\delta x = \text{tension force} \tag{6b}$$
$$\tag{6c}$$

Then, promoting the discrete index n to the continuous index x, we have

$$\ddot{y} = c^2 y'' \tag{7}$$

where the "sound speed" c is given by

$$c^2 = \frac{\kappa}{\lambda} \tag{8}$$

2.3 Part (c): Separable solutions

Guess solutions of the form

$$y(x,t) = X(x)T(t) \tag{9}$$

where $X(x)$ is purely a function of x and $T(t)$ is purely a function of t.

Then

$$\ddot{y}(x,t) = X(x)\ddot{T}(t) \tag{10a}$$
$$y''(x,t) = X''(x)T(t) \tag{10b}$$

so that

$$X(x)\ddot{T}(t) = c^2 X''(x)T(t) \tag{11}$$

We then divide through by $y = XT$ to obtain

$$\frac{\ddot{T}(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} \tag{12}$$

2.4 Part (d): Separation of variables

Since the left hand side in Equation 12 only depends on t and the right hand side only depends on x, that actually means that neither side can depend on either quantity, i.e., they are both the same constant, which we will suggestively call $-\omega^2$:

$$\frac{\ddot{T}(t)}{T(t)} = -\omega^2 \tag{13a}$$

$$c^2 \frac{X''(x)}{X(x)} = -\omega^2 \tag{13b}$$

$$\tag{13c}$$

The first equation can be rewritten as

$$\ddot{T}(t) + \omega^2 T(t) = 0 \tag{14}$$

which is a simple harmonic oscillator equation with frequency ω, i.e., its solutions are

$$T(t) = A\cos(\omega t) + B\sin(\omega t) \tag{15}$$

The second equation can be rewritten as

$$X''(x) + k^2 X(x) = 0 \tag{16}$$

which is *also* a simple harmonic oscillator equation with (spatial) frequency $k^2 \equiv \omega^2/c^2$, i.e.,

$$X(x) = C\cos(kx) + D\sin(kx) \tag{17}$$

Note that

$$\omega^2 = k^2 c^2 \tag{18}$$

2.5 Part (e): Quantization of frequencies

The general separable solution from part (d) is

$$y(x,t) = (A\cos(\omega t) + B\sin(\omega t))\,(C\cos(kx) + D\sin(kx)) \tag{19}$$

For simplicity, we can think about the solution at $t = 0$ and pick $A = 1$:

$$y(x, 0) = C\cos(kx) + D\sin(kx) \tag{20}$$

Because $y(x = 0) = 0$, this requires that $C = 0$. Then

$$y(x, 0) = D\sin(kx) \tag{21}$$

However, since $y(x = a) = 0$ also, we have

$$y(a, 0) = D\sin(kx) = 0 \tag{22}$$

which is only solved (for nonzero D) by

$$ka = \frac{\omega a}{c} = n\pi \tag{23}$$

for an integer n, which can be taken to be positive without lack of generality.

We thus see that, in this box, the only possible separable solutions allowed are those with a discrete spectrum of frequencies and wavenumbers:

$$\omega = n\pi c/a \tag{24a}$$
$$k = n\pi/a \tag{24b}$$