California Institute of Technology
Physics 1b Recitation
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Electromagnetism and Estimation
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"I measure in my palm and use my eyes to estimate amounts; a tablespoon is a full palm of dried spices." —Rachel Ray

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1 What is electromagnetism and why study it?

1.1 What can E&M explain?

Electromagnetism (or **E&M**) is one of the four fundamental forces in the universe, and perhaps the most important in our daily lives. It is *much* easier to describe what *isn't* due to E&M:

- The **gravitational force** which attracts together objects with mass is *not* E&M. E&M *cannot* explain the orbits of planets.
- The **strong force** which holds together atomic nuclei and the quarks which make up protons and neutrons is *not* E&M. E&M *cannot* explain why protons in a nucleus stick together.
- The **weak force** which transmutes different flavors of quarks between each other is *not* E&M. E&M *cannot* explain radioactivity.

Almost any other physics which does not fall into one of the three categories above *is*, in one way or another, a consequence of E&M. In other words, the behavior of any objects which are lighter than asteroids and bigger than atomic nuclei are almost certainly dominated by E&M. For example:

- Phenomena like **static electricity** and **lightning** are consequences of the electric forces both caused and experienced by electrons.
- All of our **electrical appliances** rely on a detailed understanding of how the behavior of E&M fields in conductors can direct energy in creative ways, and modern computers rely on the ability of this physics to implement logic.
- The **light** that we see is made up of oscillations in the electromagnetic field, called **electromagnetic waves**, which can be shown to travel at a speed $c = 1/\sqrt{\mu_0 \epsilon_0} \approx 3 \times 10^8\,\mathrm{m\,s^{-1}}$ (where ϵ_0 and μ_0 control the strengths of the **electric** and **magnetic forces**, respectively).
- The **normal** and **frictional forces** which appear in classical mechanics are fundamentally caused by the electric repulsion between the electron clouds of the atoms making up two contacting surfaces. The electric force is why you do not fall through the floor, and really what defines our intuitive notion of **touching** in the first place.
- Almost the entirety of the field of **chemistry** concerns the complicated consequences of electromagnetic interactions between the electrons of various atoms (when combined with quantum mechanics). E&M governs what kinds of molecules can form, what kinds of reactions they will have, and almost every property of materials.

1.2 What are the *rules* of E&M?

At a very high level, E&M describes a kind of stuff called **charge** (which can be positive and negative) and how it interacts with the **electric** and **magnetic fields**, which take on vector values everywhere in space.

The electric and magnetic fields are denoted by $\vec{E}$ and $\vec{B}$, respectively, and the charge and **current** (i.e., flow of charge) densities are denoted by ρ and $\vec{j}$, respectively.

The "rules" of electromagnetism are given by five equations. Four of these describe how the electric and magnetic fields respond to charge, and how they interact with each other. These are called

Maxwell's equations, and, in SI units, are:

- **Gauss's law:**
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

- **No magnetic monopoles:**
$$\nabla \cdot \vec{B} = 0 \tag{2}$$

- **Faraday's law:**
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

- **Ampére's law:**
$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4}$$

where $\epsilon_0 \approx 8.85 \times 10^{-12}\,\mathrm{C^2\,s^2\,kg^{-1}\,m^{-1}}$ is the **vacuum permittivity** and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kg\,m\,C^{-2}}$. These unnecessarily complicatedly-named constants roughly control how strong the electric and magnetic fields are.

The respective plain-English translations of these rules are

- The electric field *flows* out of regions where there is charge. Very roughly speaking, electric field lines "start" where there is positive charge and "end" where there is negative charge.
- The magnetic field does not flow out of anywhere, i.e., magnetic field lines do not begin or end anywhere.
- A changing magnetic field generates an electric field.
- Magnetic fields *curl* around currents, i.e., magnetic fields tend to "wrap around" the direction of moving charges. Also, a changing electric field generates a magnetic field.

The last rule describes the force experienced by charges due to the electric and magnetic fields, and is called the **Lorentz force law**:
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{5}$$

where q is the charge. In plain English, this law states that:

- The electric field accelerates positive charges along it, and negative charges against it, with the strength of the charge determining the strength of the force.
- The magnetic field accelerates *moving* charges around the magnetic field, in a way that can change their **direction**, but *not* their speed.

Some fast facts about electromagnetism, which arise out of these laws and are worth remembering on their own:

- Charge is **conserved**: while charge can be moved around, it cannot be created or destroyed.

- If charges are not moving (i.e., **electrostatics**), the electric field due to a point charge q is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \tag{6}$$

  This is called **Coulomb's law**.

- If currents are not changing (i.e., **magnetostatics**), the magnetic field due to a current I flowing along a directed length $\vec{l}$ is

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I\vec{\ell} \times \hat{r}}{r^2} \tag{7}$$

  This is called the **Biot–Savart law**, and is basically the version of Coulomb's law for magnetism.

- E&M obeys **superposition**, i.e., if two distributions of charges/currents are *added* together, then their electric and magnetic fields also *add*.

  This means that, if you want to calculate the electric and magnetic fields due to a complicated (static) configuration of charges and currents, you can simply break them up into infinitesimal point charges/currents and integrate:

$$\vec{E} = \int \mathrm{d}\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\mathrm{d}q}{r^2} \hat{r} \tag{8a}$$

$$\vec{B} = \int \mathrm{d}\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\,\mathrm{d}\vec{l} \times \hat{r}}{r^2} \tag{8b}$$

2 How and why to make order-of-magnitude estimates in physics

In **scientific notation**, a number is represented as an order-unity number $1 \leq a < 10$ multiplied by an integer power of 10, i.e., a number x can be represented as

$$x = a \times 10^n \tag{9}$$

For a concrete example, Coulomb's constant ($k \approx 8990000000\,\mathrm{N\,m^2\,C^{-2}}$) can be written in scientific notation as

$$k = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}} \tag{10}$$

Not only is this way of writing k much clearer, it breaks the number up into a multiple a and a **order of magnitude** n. If two numbers have a different a but the same n, they aren't much different: $2 \times 10^{-3}\,\mathrm{m}$ and $8 \times 10^{-3}\,\mathrm{m}$ might seem quite different, but they are both some number of millimeters, and we as humans are likely to interact with objects of these two sizes in similar ways. However, if two numbers have a different n, they are likely to be quite different: $2 \times 10^{-3}\,\mathrm{m}$ is $2\,\mathrm{mm}$, but $2 \times 10^3\,\mathrm{m}$ is $2\,\mathrm{km}$!

An **order-of-magnitude estimate** is one where the primary objective is to get the number n correct. That isn't to say that we should be careless and not *try* to get a, but we aren't overly concerned if it's a bit off. As argued before, differences in a don't make much of a difference in the *scale* of the number being estimated.

2.1 $1.5^{5.7} \approx 10$

If you multiply the number 1.5 by itself $\approx$ 5.7 times, you will get the number 10. Therefore, if you were to overestimate $\approx$ 5.7 different numbers and then multiply them together, you are overestimating the answer by an order of magnitude.

That is incredibly permissive. Overestimating a number by a factor of 1.5 is like estimating the number of fingers on a hand to be 7.5, or the average height of a person to be $8'2''$, and you would have make that sort of mistake *more than five times*, and *in the same direction*, to be off by an order of magnitude.

In reality, you are much more likely to make such errors in both directions, e.g., overestimating some numbers by a factor of 1.5 and underestimating others by the same factor. In that case, the magnitude n that you calculate will *random walk*, and the error in it will accumulate much more slowly. It turns out that, if you're making factor-of-1.5 errors in both directions, you would have to do it about $5.7^2 \approx 32$ times! *The vast majority of back-of-the-envelope calculations don't even involve multiplying that many numbers together!*

When first doing order-of-magnitude estimates, it is easy to feel uncomfortable that you might be accumulating errors too quickly for your final answer to be meaningful. In reality, as long as you use common sense, take important factors into account, and have some knowledge of the scales of the physics you are trying to use, *it is hard to be very far off the real answer.*

2.2 Example 1: Sedov-Taylor explosions

We will estimate the energy E released by the Trinity nuclear test using only the picture shown in Figure 1 and some common sense.

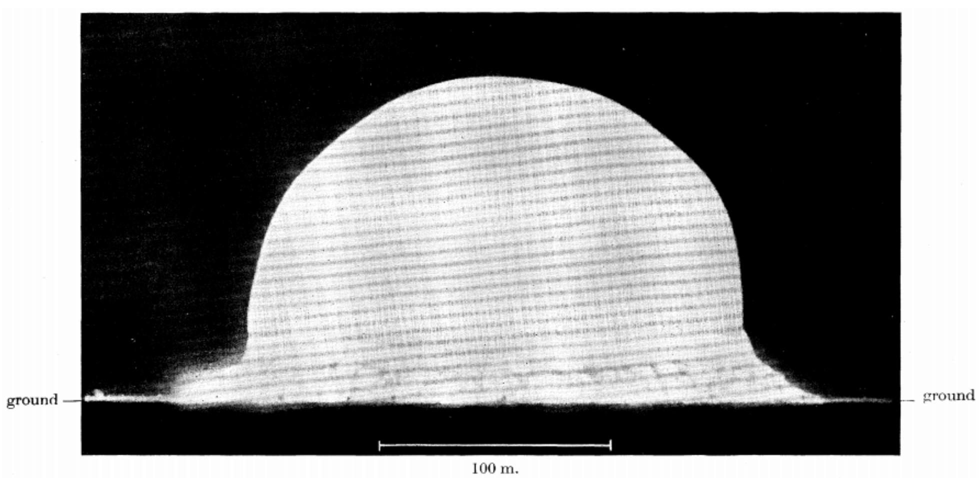

FIGURE 7. The ball of fire at $t = 15$ msec., showing the sharpness of its edge.

Figure 1: Snapshot of the Trinity nuclear test explosion at $t = 15\,\mathrm{ms}$

As the explosion expands, it sweeps up the air (which has a density $\rho \approx 1\,\mathrm{kg\,m^{-3}}$). At a time $t = 15\,\mathrm{ms}$, the explosion in the picture seems to have a radius $R \approx 100\,\mathrm{m}$.

Therefore, very roughly speaking, the explosion at this time has swept up a total mass M of air

$$M \approx \frac{1}{2}\rho \cdot \frac{4}{3}\pi R^3 \approx 2\rho R^3 \tag{11}$$

The velocity of the shock front is $v \sim R/t$. Therefore, the total kinetic energy in the engulfed material (which should be $\sim E$) can be estimated as

$$E \sim \frac{1}{2}Mv^2 \sim \rho R^5/t^2 \sim 10\,\text{kilotons of TNT} \tag{12}$$

where a kiloton of TNT is $\sim 4.2 \times 10^{12}\,\text{J} = 4.2\,\text{TJ}$. In fact, this is within a factor of 2 to the official estimate for the yield of the explosion. More recently, the yield has been re-estimated as $E \sim 25\,\text{kilotons of TNT}$[1]. Perhaps the factor of two may be accounted for by another half of the energy going into the ground.

Even though we weren't particular concerned about it, we still got a pretty good estimate not just for n but an alright one for a!

2.3 Example 2: A "special" radius in general relativity

The constant that governs the strength of gravity is $G \approx 6.67 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2}$, and the speed of light is $c \approx 3 \times 10^8\,\text{m}$.

It is natural to wonder if, given some object of mass M, there is some "special" length scale that arises from gravity and the speed of light both being relevant.

The only way to combine these three variables G, c, and M (modulo order-unity factors) is

$$R \sim \frac{GM}{c^2} \tag{13}$$

This is, within a factor of two, the radius of a black hole with a mass M. However, using just dimensional analysis, we were able to guess that *something* was going to happen at this scale, even if we weren't really sure what it was going to be.

2.4 Example 3: The significance of wrong answers

Planck's constant $\hbar \approx 1.06 \times 10^{-34}\,\text{J}\,\text{s}$ appears when quantum mechanics is important. Coulomb's constant k appears when electromagnetism is important. Electrons and protons both have a charge $e \approx 1.60 \times 10^{-19}\,\text{C}$. Finally, electrons have masses $m_e \approx 9.11 \times 10^{-31}\,\text{kg}$.

There is only one way to combine these variables together into a length:

$$a_0 \sim \frac{\hbar^2}{ke^2 m_e} \tag{14}$$

[1]Selby, H. D., Hanson, S. K., Meininger, D., Oldham, W. J., Kinman, W. S., Miller, J. L., ... & Marcy, P. W. (2021). A New Yield Assessment for the Trinity Nuclear Test, 75 Years Later. *Nuclear Technology*, 207(sup1), 321-325.

This length scale R is called a **Bohr radius**, and is roughly the length scale of an electron cloud.

Suppose we were to try to do the same thing, but for the nucleus, which we know to be made up of protons and neutrons. Let's ignore the fact that we know that protons repel each other, and see if a similar estimate is possible for the radius of a nucleus. It seems like the only thing we should need to do is replace m_e with $m_p \approx 1.67 \times 10^{-27}$ kg (the mass of a proton).

This gives a length scale

$$R \sim \frac{\hbar^2}{ke^2 m_p} \sim 6 \times 10^{-4}\, a_0 \tag{15}$$

In reality, atomic nuclei have radii $R \sim 10^{-5} a_0$, which is different from our estimate by a factor of 60! Did we do something wrong?

The fact that our **naïve** estimate is wrong is not purely negative, but (once arithmetic errors have been ruled out) demands an explanation! In this case, the reason why our estimate was so far off was because, for atomic nuclei, the strong force becomes very important, and, because we did not know about it, we did not incorporate the characteristic scale of that force into our estimate.

The lesson is that, if order-of-magnitude estimates are wrong, *try to explain them*! Don't just write it off as a natural consequence of the haphazard nature of your approximations—order-of-magnitude estimates are more accurate than you might think!

Incorrect estimates are often a reason to be *excited*: it means that there is something fun to explain, and possibly even new, as-of-yet undiscovered physics! If an observed number defies order-of-magnitude estimation, it might indicate that something is being left out of our current understanding of physics. Roughly this sort of logic underlies many modern-day open physics problems (e.g., the **hierarchy** and **vacuum catastrophe problems**).