A Sketch of the LIGO Detectors

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Table of Contents

What this Talk Covers (and Some of the Things it Doesn't)

1. This talk aims to go past the classic Michelson interferometer picture, following Bond et al. [2]
2. Accordingly, it includes (brief) introductions to cavities, to detection schemes, and to noise sources
3. It *does not* include a number of things that are really important to understand the detectors, most notably any discussion of control systems or mode cleaning

Gravitational Waves in Brief

1. Gravitational waves originating from the quadrupolar moment are given by [1]:

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2. G/c^4 is very small, but scaling with luminosity distance is inverse, instead of inverse square!
3. For L shaped detectors, we want to measure:

$$h = \frac{L_X - L_Y}{L} = F_+h_+ + F_\times h_\times \sim O(10^{-21}) \tag{2}$$

Table of Contents

Optical Elements

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3. We will be particularly interested in how this interacts with mirrors, and propagates through space

Mirrors

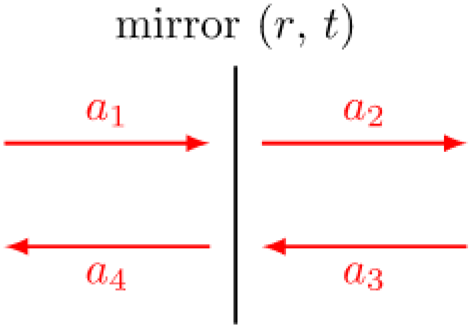

Figure: An illustration of the idealized mirror

Mirrors

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2. And rearrange to a corresponding matrix equation

$$\begin{pmatrix} a_1 \\ a_4 \end{pmatrix} = \frac{i}{t} \begin{pmatrix} -1 & r \\ -r & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} \tag{6}$$

Empty Space

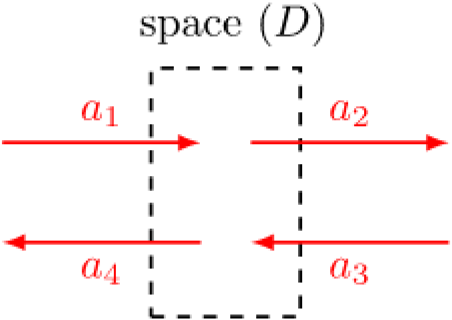

Figure: An illustration of scalar wave propagation through space

Empty Space

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$$\begin{pmatrix} a_1 \\ a_4 \end{pmatrix} = \begin{pmatrix} \exp(ikD) & 0 \\ 0 & \exp(-ikD) \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} \tag{9}$$

Two Mirror Systems

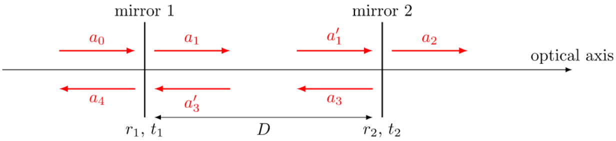

Figure: An illustration of the idealized two mirror system

Two Mirror Systems

1. Now we can make use of the matrix convenience:

$$\begin{aligned}\begin{pmatrix} a_0 \\ a_4 \end{pmatrix} &= \frac{i}{t_1}\begin{pmatrix} -1 & r_1 \\ -r_1 & 1 \end{pmatrix}\begin{pmatrix} \exp(ikD) & 0 \\ 0 & \exp(-ikD) \end{pmatrix} \\ &\quad \times \frac{i}{t_2}\begin{pmatrix} -1 & r_2 \\ -r_2 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ 0 \end{pmatrix} \\ &= \frac{-1}{t_1 t_2}\begin{pmatrix} e^+ - r_1 r_2 e^- & -r_2 e^+ + r_1 e^- \\ -r_2 e^+ + r_1 e^- & e^- - r_1 r_2 e^+ \end{pmatrix}\begin{pmatrix} a_2 \\ 0 \end{pmatrix} \qquad (10)\end{aligned}$$

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$$= \frac{-1}{t_1 t_2} \begin{pmatrix} e^+ - r_1 r_2 e^- & -r_2 e^+ + r_1 e^- \\ -r_2 e^+ + r_1 e^- & e^- - r_1 r_2 e^+ \end{pmatrix} \begin{pmatrix} a_2 \\ 0 \end{pmatrix} \quad (10)$$

2. And solve for the ratio

$$\frac{a_2}{a_0} = \frac{-t_1 t_2 \exp(-ikD)}{1 - r_1 r_2 \exp(-i2kD)} \quad (11)$$

Properties of Fabry-Pérot Cavities

1. We can define *free spectral range* (FSR):

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3. Which together give us the cavity's finesse (F):

$$F = \frac{\mathrm{FSR}}{\mathrm{FWHM}} = \frac{\pi}{2\arcsin\left(\frac{1-r_1 r_2}{2\sqrt{r_1 r_2}}\right)} \approx \frac{\pi}{1 - r_1 r_2} \tag{14}$$

where the final approximation assumes high finesse ($r_1, r_2 \sim 1$)

Properties of Fabry-Pérot Cavities

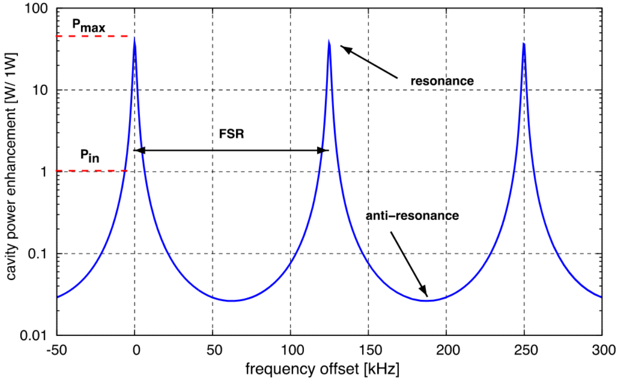

Figure: The frequency dependent power enhancement of an example Fabry-Pérot interferometer

Michelson Interferometers

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2. A beam of light is split, sent along perpendicular arms, then recombined.
3. Any difference in distance traveled (modulo the wavelength of the light) results in a phase difference, such that when recombined the beams destructively interfere.

Michelson Interferometer Optics

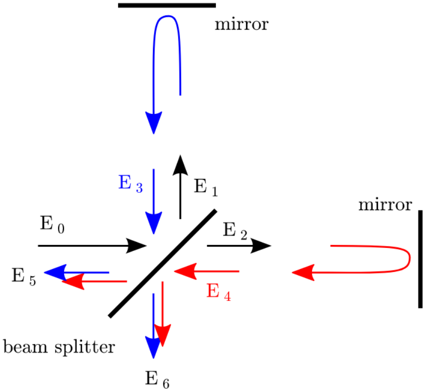

Figure: A schematics of optical fields in a simple Michelson interferometer

Michelson Interferometer Optics

1. The outgoing field will be:

$$E_6 = E_0 rt\left[\exp\left(i(\phi_t+\phi_{r1}+\Phi_1)\right)+\exp\left(i(\phi_t+\phi_{r2}+\Phi_2)\right)\right] \tag{15}$$

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2. According to convention $\phi_t = \pi/2$, and $\phi_{r1} = \phi_{r2} = 0$.
3. So we can simplify and express in terms of common and differential phase:

$$E_6 = irtE_0 \exp\left(i\frac{\Phi_1+\Phi_2}{2}\right)2\cos\left(\frac{\Phi_1-\Phi_2}{2}\right) \tag{16}$$

CARM and DARM in Michelson Interferometers

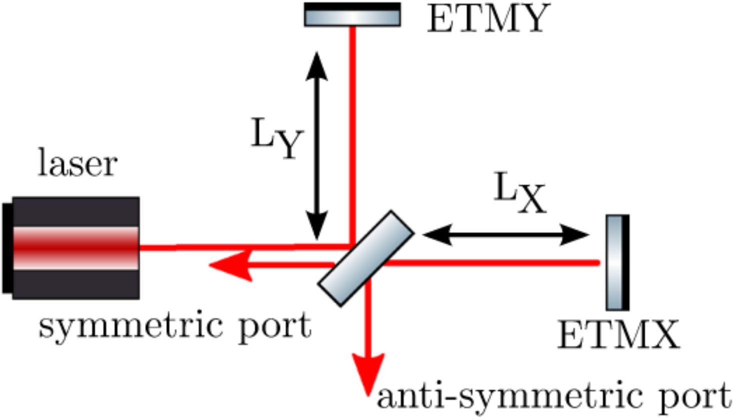

Figure: A schematic of the prototypical Michelson interferometer

CARM and DARM in Michelson Interferometers

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3. Power depends only on DARM, not on the lengths themselves, and has minima known as *dark fringes*

Table of Contents

Phase Modulation

1. Phase modulations are signals on top of the carrier which look like:

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$$E \approx E_0 \exp(i\omega_0 t) \left[1 - \frac{m^2}{4} + i\frac{m}{2}\Big(\exp(-i\Omega t) + \exp(i\Omega t) \Big) \right] \tag{20}$$

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3. Importantly, GWs manifest themselves in the detectors as a phase modulation on the carrier beam, giving rise to *signal sidebands*

Signal Sidebands

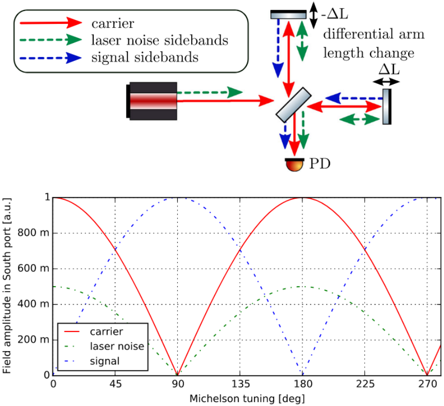

Figure: A schematic of sidebands in a Michelson inteferometer, and the effect of tuning on the carrier and the sidebands

Homodyne Detection

1. If we sit at exactly the dark fringe, the signal sidebands will have nothing to beat against. LIGO handles this with the homodyne detection scheme, introducing a small DC offset δ_{off} from the dark fringe:

$$\Delta L = \frac{\pi}{2k_0} + \delta_{off} \tag{21}$$

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2. Plugging this in along with our GW induced sidebands gives:

$$\begin{aligned} E &= irtE_0 \exp(i2k_0\bar{L})\exp(i\omega_0 t)(2\cos(k_0\Delta L) + s^+ + s^-) \\ &= irtE_0 \exp(i2k_0\bar{L})\exp(i\omega_0 t)(2\sin(k_0\delta_{off}) + s^+ + s^-) \end{aligned} \tag{22}$$

Homodyne Detection Signal Power

1. Solving for the power gives

$$P = EE^* = TR|E_0|^2 \Big(4 \sin^2(k_0 \delta_{off}) + 2 \sin(k_0 \delta_{off})(s^+ + s^-) + O(s^2) \Big) \tag{23}$$

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2. Notably, we need this DC offset for the sidebands to be visible at all. Control systems are used to maintain the interferometer in this state.

A Simple GW

1. Consider a simplified gravitational wave

$$h(t) = h_0 \cos(\omega_{gw} t + \phi_{gw}) \tag{24}$$

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2. We can compute the phase shift due to the gravitational wave with:

$$\begin{aligned} \phi &= -k_0 L \mp \frac{\omega_0}{2} \int_{t-L/c}^{t} h(t) = -k_0 L \mp \delta\phi \\ \delta\phi &= \frac{\omega_0 h_0}{\omega_{gw}} \cos\left(\omega_{gw} + \phi_{gw} - \frac{\omega_{gw} L}{2c}\right) \sin\left(\frac{\omega_{gw} L}{2c}\right) \end{aligned} \tag{25}$$

Sidebands due to a GW

1. We can identify the elements of phase modulation within the equation for $\delta\phi$:

$$\begin{aligned} m_{gw} &= -\frac{\omega_0 h_0}{\omega_{gw}} \sin\left(\frac{k_{gw} L}{c}\right) \\ \Phi &= -\frac{k_{gw} L}{2} + \phi_{gw} \end{aligned} \tag{26}$$

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2. Which allows one to work work out terms from the sideband equation:

$$\begin{aligned} A_{gw} &= \frac{m_{gw}}{2} = -\frac{\omega_0 h_0}{2\omega_{gw}} \sin\left(\frac{k_{gw} L}{c}\right) \\ \Phi^{\pm}_{gw} &= \frac{\pi}{2} - Lk_0 \pm \left(-\frac{k_{gw} L}{2} + \phi_{gw}\right) \\ \alpha^{\pm}_{gw} &= A_{gw} \exp(i\Phi^{\pm}_{gw}) \exp(\pm i\omega_{gw} t) \end{aligned} \tag{27}$$

Sidebands Coupling

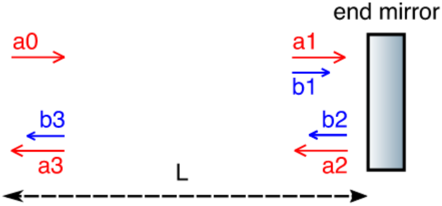

Figure: A schematic of the propagation of the carrier beam and the signal sidebands in one interferometer arm

Sidebands Coupling Mathematics

1. Proceeding, the carrier beams satisfy:

$$\begin{aligned} a_3 &= a_2 \exp(-ik_0 L) \\ a_2 &= r_{etm} a_1 \\ a_1 &= a_0 \exp(-ik_0 L) \end{aligned} \tag{28}$$

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2. And the sidebands satisfy:

$$\begin{aligned} b_1^\pm &= a_0 \alpha_{gw}^\pm \\ b_2^\pm &= r_{etm} b_1^\pm \\ b_3^\pm &= b_2^\pm \exp(-i(k_0 \pm k_{gw})L) + a_2 \alpha_{gw}^\pm \end{aligned} \tag{29}$$

More Sidebands Coupling Mathematics

- This works out to give

$$\begin{aligned} b_3^{\pm} &= 2 r_{etm} a_0 \alpha_{gw}^{\pm} \exp\left(- i k_0 L \mp i \frac{k_{gw} L}{2} \right) \\ &\times \cos\left(\mp i \frac{k_{gw} L}{2} \right) \\ &= -i \frac{r_{etm} a_0 \omega_0 h_0}{2\omega_{gw}} \sin(k_{gw} L) \exp(-i 2 k_0 L) \\ &\times \exp(\pm i(\omega_{gw} t - k_{gw} L + \phi_{gw})) \end{aligned} \tag{30}$$

Power on the Photodiode

1. The equation for output field is

$$E_{out} = i2rtE_0 \cos(k_0 \Delta L) + b_X^+ + b_X^- + b_Y^+ + b_Y^- \tag{31}$$

where b_X and b_Y are the X and Y arm sidebands respectively.

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   where b_X and b_Y are the X and Y arm sidebands respectively.

2. After they recombine, these end up taking the same form, except that the X arm takes a negative sign (the kludge for the antenna response), and they have respective lengths plugged in. For conciseness, I will not write this out, nor the substitution of CARM and DARM.

Power on the Photodiode

1. After doing the summing and some simplification, we can get out the expression for the field:

$$E_{out} = iE_0 \cos(k_0 \Delta L) - i\frac{2k_0 \delta_{off} E_0 \omega_0 h_0}{\omega_{gw}} \sin(k_{gw}\bar{L}) \cos(\omega_{gw} t - k_{gw}\bar{L} + \phi_{gw}) \tag{32}$$

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2. This in turn lets us compute the power due to the GW:

$$P_{gw} \approx 2k_0 \delta_{off} \frac{\omega_0 h_0}{\omega_{gw}} |E_0|^2 \sin(k_{gw} L) \cos(\omega_{gw} t - k_{gw} \bar{L} + \phi_{gw}) \quad (33)$$

and the transfer to the output photodiode:

$$T_{gw \to P}(\omega_{gw}) \approx k_0 \delta_{off} |E_0|^2 \frac{\omega_0}{\omega_{gw}} \sin(k_{gw} \bar{L}) \exp(-ik_{gw} \bar{L}) \quad (34)$$

Table of Contents

Shot Noise

1. The PSD of shot noise for a DC offset scheme is:

$$S_{P,DC} \approx 2P_0(k_0\delta_{off})^2\hbar\omega_0 \tag{35}$$

And so using the transfer function given previously, one may compute the noise to signal ratio due to shot noise as:

$$NSR = \frac{\sqrt{S_{P,DC}}}{T_{gw\to P}} = \sqrt{\frac{2\hbar}{P_0\omega_0}}\frac{\omega_{gw}}{\sin(\omega_{gw}\bar{L}/c)}\frac{h}{\sqrt{Hz}} \tag{36}$$

where the unit is the slightly odd strain per root Hz.

Shot Noise

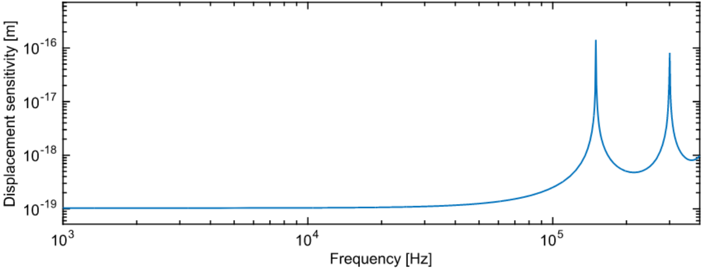

Figure: Shot noise sensitivity limit for $P_0 = 1\ W$ and $L = 1\ km$

Radiation Pressure

1. Radiation pressure is another fundamental noise source, that of vacuum noise coupling to the mirrors, which results in a power spectral density:

$$S_{\phi,RP} = \frac{8\hbar P_0^3 \omega_0 k_0^2}{M^2 c^2 \Omega^4} \tag{37}$$

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2. So the NSR due to this will be

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3. Notably this *increases* with the square root of power, but it decreases with the square of the frequency, and so is principally a low frequency noise source.

Full Quantum Noise Sensitivity

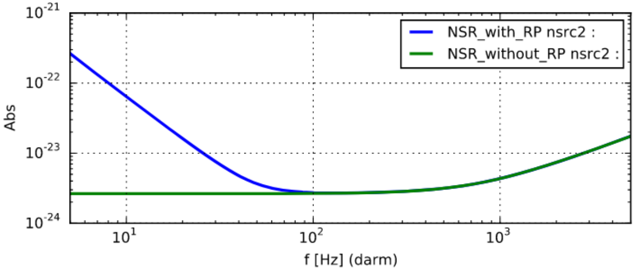

Figure: Sensitivity limits due to both shot noise and radiation pressure

Full aLIGO Noise Breakdown

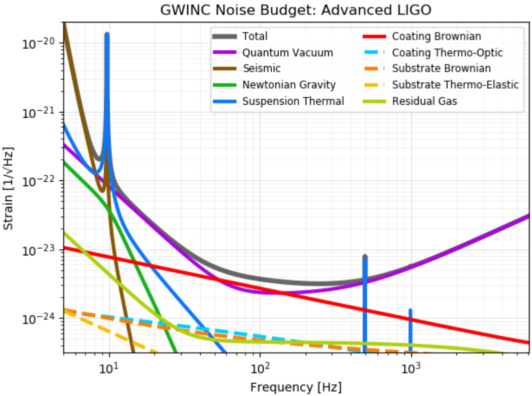

Figure: Breakdown of noise sources in aLigo according to the GWINC model[3]

Table of Contents

Power Recycling

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2. Adding a power recycling cavity (a Fabry-Pérot cavity before the beam splitter) provides gain proportional to the finesse of the cavity:

$$G_{PR} \approx \frac{\mathcal{F}}{\pi} \tag{39}$$

Power Recycling

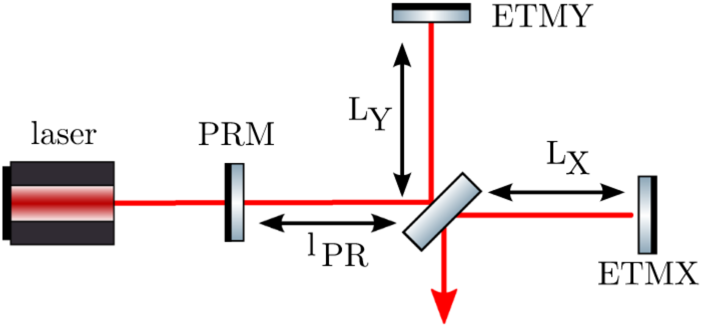

Figure: The layout of an interferometer with a power recycling cavity added

Arm Cavities

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2. Moreover, they increase the effective length of the detector arms, which massively boosts sensitivity

Arm Cavities

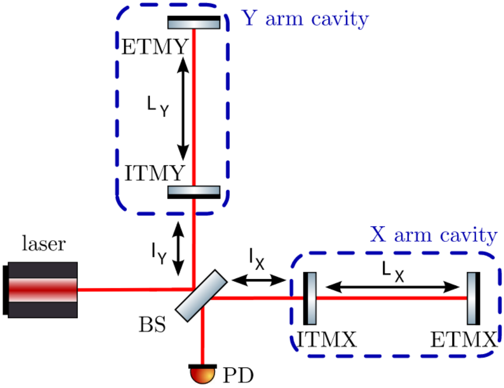

Figure: The layout of an interferometer with arm cavities added

Signal Recycling

1. A signal recycling cavity is a Fabry-Pérot cavity placed at the output port, which allows for resonant sideband extraction.
2. If the finesse of arm cavities above is very high, then a very sharp resonant feature is developed, narrowing the bandwidth.
3. The SRC is tuned to an *anti*-resonant operating point, which increases the bandwidth of the detector.

Signal Recycling

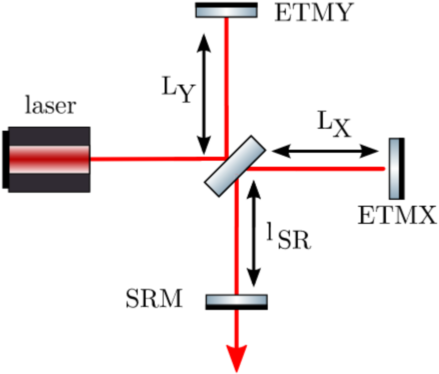

Figure: The layout of an interferometer with a signal recycling cavity added

Effect of tuning

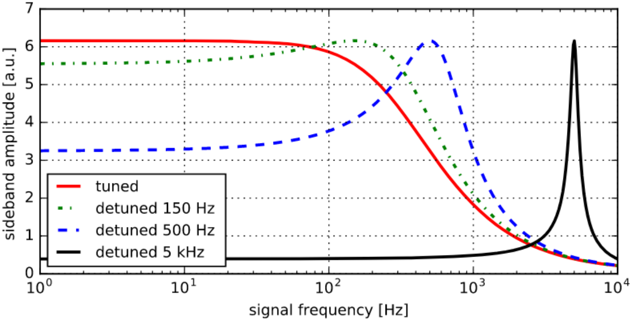

Figure: Sideband amplitude for various tunings of the SRC

Conclusions

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2. Careful controls allow for much higher sensitivity than is otherwise possible.
3. Considerations of maximizing sensitivity in frequency domains of interest determine the design and operation of the detector, and require constant management of trade-offs.