California Institute of Technology
Physics 1b Recitation
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"Ben Franklin may have discovered electricity, but it is the man who invented the meter who made the money." —Earl Warren

Contents

1 Practice Problem

Consider the circuit shown in Figure 1:

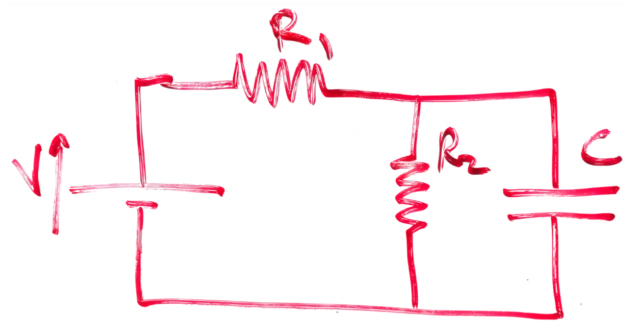

Figure 1: The circuit described in the practice problem

(a) Find the charge on the capacitor, C, in steady state.

(b) Suppose the resistor R_1 is replaced by a wire. In steady state, how much current flows through R_2?

(c) Suppose the resistor R_2 is removed. In steady state, how much current flows through R_1?

(d) Suppose the capacitor originally has no charge on it, $Q_c(t = 0) = 0$. Write and solve a differential equation for Q_c.

(e) What is the timescale over which the capacitor charges up?

(f) At $t \to \infty$, what is the charge on the capacitor? Does that make sense?

(g) Suppose the capacitor (with capacitance C) is a parallel plate capacitor with vacuum in between the plates. Now consider filling up the entire space between the plates with a material with dielectric constant κ.

How does Q_c evolve over time, assuming again that it starts uncharged? Over what timescale does this happen?

(h) **[Extra]** Now replace the battery with a voltage source which oscillates over time with angular frequency ω and amplitude V_0, i.e.,

$$V(t) = V_0 \cos \omega t \tag{1}$$

Find the charge on the capacitor with time, assuming that it starts with a charge $Q_c(0) = Q_{0c}$.

Hint: It is helpful to work with complex numbers, and represent the voltage as the real part of $V(t) = V_0 e^{i\omega t}$, using the fact that $e^{i\omega t} = \cos \omega t + i \sin \omega t$.

2 Problem Solution

2.1 Part (a): Steady-state capacitor charge

In steady state, the capacitor is fully charged up, and admits no current at all. Therefore, the current can only pass through the resistors, and the effective resistance of the resistors (which are in series) is $R_{\text{eff}} = R_1 + R_2$. Therefore, the current passing through them is, by Ohm's law,

$$I = V/R_{\text{eff}} = V/(R_1 + R_2) \tag{2}$$

By Ohm's law again, the voltage drop across R_2 is

$$V_2 = IR_2 = \frac{R_2}{R_1 + R_2} V \tag{3}$$

Finally, the voltage drop across R_2 must be identical to the voltage drop across the capacitor, since they are in parallel. Therefore, the charge Q_c on the capacitor is

$$Q_c = CV = \frac{R_2 C}{R_1 + R_2} V \tag{4}$$

2.2 Part (b): Steady-state R_2 current (without R_1)

If R_1 is replaced by a wire, then the voltage drop across R_2 is simply V, and its current is simply given by Ohm's law:

$$I_2 = V/R_2 \tag{5}$$

2.3 Part (c): Steady-state R_1 current (without R_2)

If R_2 is removed entirely, then the resistor R_1 is in series with the capacitor. In steady state, no current may flow through the capacitor, and the current through R_1 is $I_1 = 0$ (since their currents must match).

2.4 Part (d): Charging the capacitor

We can use Ohm's law to write down equations for the two resistors, in terms of their voltage drops and currents:

$$V_1 = I_1 R_1 \tag{6a}$$
$$V_2 = I_2 R_2 \tag{6b}$$

Furthermore, the voltages should add to V:

$$V = V_1 + V_2 \tag{7}$$

If the charge on the capacitor is Q_c, then

$$I_1 = I_2 + \dot{Q}_c \tag{8}$$

where $\dot{Q}_c$ denotes $\mathrm{d}Q_c/\mathrm{d}t$.

Finally, the capacitor also has a voltage drop V_2, and therefore

$$V_2 = Q_c/C \tag{9}$$

We can start with Equation 7, and substitute in the expressions for voltage, arbitrarily picking both resistors:

$$V = I_1 R_1 + I_2 R_2 \tag{10}$$

Equations 10 and 8 are sufficient to eliminate one of the currents. We can eliminate I_1:

$$V = (I_2 + \dot{Q}_c)R_1 + I_2 R_2 = \dot{Q}_c R_1 + I_2(R_1 + R_2) \tag{11}$$

Finally, we can use the fact that the voltage difference across the capacitor and resistor R_2 are the same to eliminate I_2:

$$I_2 R_2 = Q_c/C \tag{12}$$

so that

$$I_2 = Q_c/R_2 C \tag{13}$$

We therefore have

$$V = \dot{Q}_c R_1 + \frac{R_1 + R_2}{R_2 C} Q_c \tag{14}$$

This can be rewritten as the differential equation

$$\frac{\mathrm{d}Q_c}{\mathrm{d}t} = \frac{V}{R_1} - \frac{R_1 + R_2}{R_1 R_2 C} Q_c = \frac{R_1 + R_2}{R_1 R_2 C}\left(\frac{R_2 C}{R_1 + R_2} V - Q_c\right) \tag{15}$$

or, separating variables

$$\int_0^{Q_c} \frac{\mathrm{d}Q_c'}{\frac{R_2 C}{R_1 + R_2} V - Q_c'} = \int_0^t \frac{R_1 + R_2}{R_1 R_2 C}\,\mathrm{d}t' \tag{16}$$

This evaluates to

$$\ln\left(\frac{R_2 C V/(R_1 + R_2)}{R_2 C V/(R_1 + R_2) - Q_c}\right) = \frac{R_1 + R_2}{R_1 R_2 C} t \tag{17}$$

so that

$$\frac{1}{1 - Q_c(R_1 + R_2)/R_2 C V} = \exp\left(\frac{R_1 + R_2}{R_1 R_2 C} t\right) \tag{18}$$

which can be solved for Q_c:

$$Q_c(t) = \frac{R_2 C}{R_1 + R_2} V \left[1 - \exp\left(-\frac{R_1 + R_2}{R_1 R_2 C} t\right)\right] \tag{19}$$

2.5 Part (e): Charging timescale

We see that t only appears in Equation 19 together with other variables in the combination

$$\frac{t}{\tau} = \frac{R_1 + R_2}{R_1 R_2 C} t \tag{20}$$

which defines a timescale (or, in this case, "time constant")

$$\tau = \frac{R_1 R_2 C}{R_1 + R_2} \tag{21}$$

2.6 Part (f): Final capacitor charge

We see that, when $t \to \infty$, Equation 19 gives

$$Q_c(\infty) = \frac{R_2 C}{R_1 + R_2} V \tag{22}$$

which agrees with our answer from part (a).

2.7 Part (g): Adding a dielectric

A dielectric simply increases the capacitance by a factor κ. We can therefore replace all instances of C with κC:

$$Q_c(t) = \frac{\kappa R_2 C}{R_1 + R_2} V \left[1 - \exp\left(-\frac{R_1 + R_2}{R_1 R_2 \kappa C} t\right)\right] \tag{23}$$

and

$$\tau = \frac{\kappa R_1 R_2 C}{R_1 + R_2} \tag{24}$$

We see that this change has two effects: it increases the maximum charge that the capacitor stores, and it makes it take longer to charge up.

2.8 Part (h): Alternating current

Everything is more or less the same, except we can replace V (which was a constant before) by $V_0 e^{i\omega t}$ in Equation 15:

$$\frac{\mathrm{d}Q_c}{\mathrm{d}t} = \frac{V_0}{R_1} e^{i\omega t} - \frac{R_1 + R_2}{R_1 R_2 C} Q_c \tag{25}$$

We can do this because

$$e^{i\omega t} = \cos \omega t + i \sin \omega t \tag{26}$$

and we would like to do this because taking the derivative of exponentials is simple. We can simply take the real part afterwards to recover Q_c.

Because the voltage oscillates with frequency ω, we can guess that Q_c must do the same:

$$Q_c = Q_{c0} e^{i\omega t} \tag{27}$$

so that

$$\frac{\mathrm{d}Q_c}{\mathrm{d}t} = i\omega Q_{c0} e^{i\omega t} \tag{28}$$

and

$$i\omega Q_{c0} = \frac{V_0}{R_1} - \frac{R_1 + R_2}{R_1 R_2 C} Q_{c0} \equiv V_0/R_1 - Q_c/\tau \tag{29}$$

where, conveniently, $e^{i\omega t}$ now appears in all equations, and can be divided out. We can then straightforwardly solve for Q_{c0}:

$$Q_{c0} = \frac{V_0}{R_1(i\omega + 1/\tau)} = \frac{V_0(1/\tau - i\omega)}{R_1(\omega^2 + 1/\tau^2)} \tag{30}$$

Therefore, the charge on the capacitor as a function of time is

$$Q_c = \left(\frac{V_0(1/\tau - i\omega)}{R_1(\omega^2 + 1/\tau^2)} \right) (\cos \omega t + i \sin \omega t) \tag{31}$$

or

$$Q_c = \frac{V_0}{R_1(\omega^2 + 1/\tau^2)} \left(\frac{1}{\tau} \cos \omega t + \omega \sin \omega t \right) + i \frac{V_0}{R_1(\omega^2 + 1/\tau^2)} \left(\frac{1}{\tau} \sin \omega t - \omega \cos \omega t \right) \tag{32}$$

Taking the real part, we have

$$Q_c = \frac{V_0}{R_1(\omega^2 + 1/\tau^2)} \left(\frac{1}{\tau} \cos \omega t + \omega \sin \omega t \right) \tag{33}$$

Note: Since Ohm's law becomes $V = IR = \dot{Q}R = i\omega QR$ in oscillating (i.e., alternating) current and $V = Q/C$, a capacitor is sometimes said to have a complex version of a resistance called an impedance $Z_C = 1/i\omega C$. Working with impedances greatly simplifies calculations by turning differential equations into algebraic ones, but at the cost of requiring complex numbers (which is, in the grand scheme of things, a small cost).