California Institute of Technology
Physics 1b Recitation
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Magnetic Personality
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"Your mind is a magnet. You don't attract what you need or what you want; you attract who you are. And I love who I am!" —Carlos Santana

Contents

1 Comparing the electric and magnetic fields

The **electric field** $\vec{E}$ and **magnetic field** $\vec{B}$ at first seem quite different, but their calculations are spiritually very similar. This section will present some of these similarities in the standard framework, where we assume that there are no magnetic monopoles.

To calculate the electric field in **electrostatics** (with stationary charges and fields), we use **Gauss's law** (which describes how electric fields are related to charges):

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = Q_{\text{enc}}/\epsilon_0 \tag{1}$$

where the integral is over a closed *surface*.

Similarly, to calculate the magnetic field in **magnetostatics** (with steady currents and fields), we use **Ampére's law** (which describes how magnetic fields are related to currents):

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\text{enc}} \tag{2}$$

where the integral is over a closed *loop*.

The only real differences are that the integral is done over a one-dimensional *closed* loop (where the enclosed currents I_{enc} are those which flow "through" the loop), and that the "electric" constant ϵ_0 is replaced by the "magnetic" constant $1/\mu_0$. Note that the integral should be done in a counterclockwise direction relative to *positive* current flowing through it.

In electrostatics, **Coulomb's law** gives the electric field due to a small volume of charge with charge density ρ:

$$\mathrm{d}\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho\,\mathrm{d}V\,\hat{r}}{r^2} \tag{3}$$

In magnetostatics, the **Biot–Savart law** gives the magnetic field due to a small segment of current with directed length $\mathrm{d}\vec{l}$:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{l} \times \hat{r}}{r^2} \tag{4}$$

The electric field produces a force on objects with charge q:

$$\vec{F} = q\vec{E} \tag{5}$$

The magnetic field produces a force which carry segments of current $I\vec{l}$:

$$\vec{F} = I\vec{l} \times \vec{B} \tag{6}$$

Since a single particle of charge q moving at a velocity $\vec{v} = \mathrm{d}\vec{l}/\mathrm{d}t$ carries a current $q/\mathrm{d}t$ through a length $d\vec{l}$, this can be written for a point charge as

$$\vec{F} = q\vec{v} \times \vec{B} \tag{7}$$

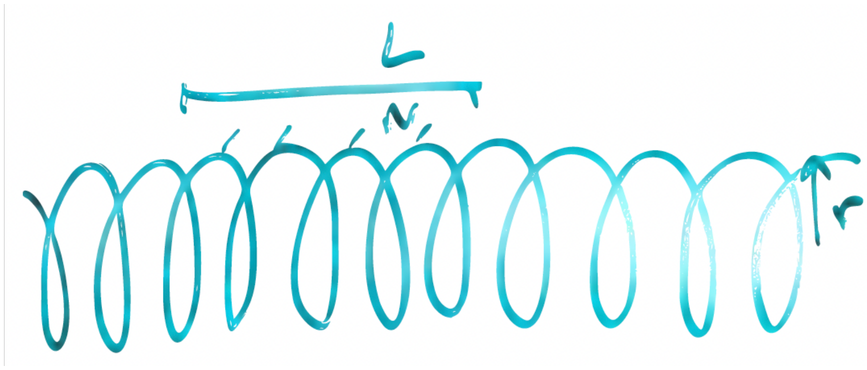

Figure 1: A prototypical solenoid

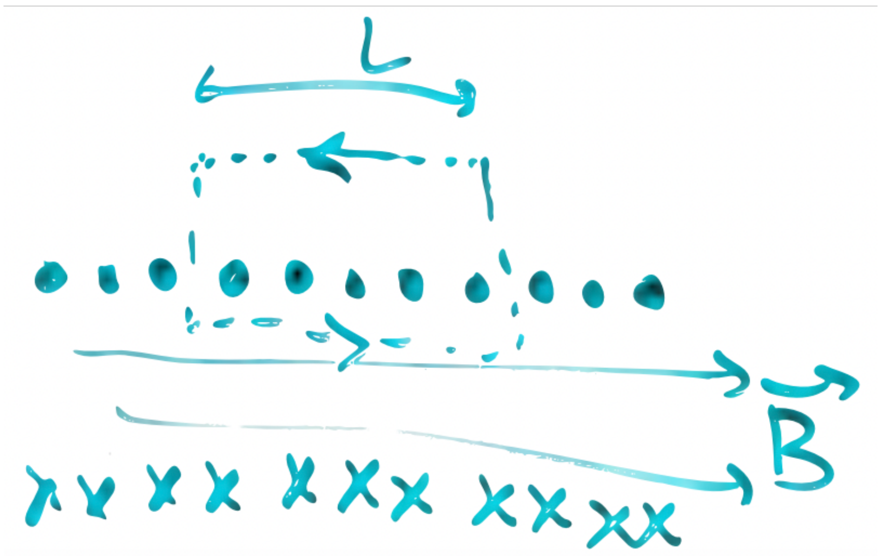

Figure 2: A convenient "Ampérian" loop

In electrostatics, a canonical problem is to find the electric field due to two (infinite) parallel plates with some surface charge density $\sigma = \pm Q/A$, yielding an electric field $E = \sigma/\epsilon_0 = Q/A\epsilon_0$. This is the prototypical **capacitor**, a foundational linear circuit device for which $V \propto Q$. They store energy in the electric field produced by the charges that they carry.

Similarly, in magnetostatics, the canonical problem is to find the magnetic field in a **solenoid**, a tightly wound loop of wire, carrying a current I. Suppose that a solenoid is infinite, with a number of loops per unit length N/L, and radius r (Figure 1). We will assume there is a large number density of loops, so that the motion of the charges along the direction of the solenoid are negligible.

We can perform this calculation by drawing a rectangular wire loop parallel to the solenoid, with one end in and one end out (Figure 2, where currents pointing out of the page are denoted by "$\cdot$" and currents pointing in the page are denoted by "$\times$"). Denote the length of the loop along the direction of the solenoid as L.

We can argue by symmetry that there should not be any radial magnetic field in the following way: the effect of turning the solenoid by 180° should be equivalent to reversing the current, but the latter flips the radial component of the field while the latter does not. Hence the magnetic field must only point along the solenoid (and must only be a function of radius). Moreover, far away from the solenoid, there should not be any field produced (since, to a faraway observer, there is no charge going anywhere). Drawing an Ampérian loop like the one shown in Figure 2 but with one side at infinity and another outside the solenoid (which encloses no currents) reveals that the field along each leg should be the same: if the magnetic field vanishes far away, it must also vanish *everywhere outside* the solenoid.

Hence, the integral part of Ampére's law becomes

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = BL \tag{8}$$

The enclosed current is simply

$$I_{\text{enc}} = (N/L) \times LI = NI \tag{9}$$

Using Ampére's law then gives

$$\vec{B} = \mu_0 I(N/L)\hat{z} \tag{10}$$

where $\hat{z}$ is the direction along the wire, relative to which the current flows counterclockwise.

This is the prototypical **inductor**, a foundational linear circuit device for which $V \propto \ddot{Q}$. They store energy in the magnetic field produced by the charges that they carry.

2 Magnetic monopoles

The analogies between the electric and magnetic fields is strengthened if one assumes the existence of **magnetic monopoles**, objects with the property that magnetic field lines can flow into or out of them. Magnetic monopoles have never been definitively[1] detected, although they are predicted by some grand unified theories which assert that they ought to a generic consequence of some early-universe phase transitions in some fundamental quantum fields[2].

However, when magnetic monopoles are introduced, the equations look exactly symmetric between the electric and magnetic fields. Problems involving magnetic fields may become conceptually or computationally more tractable (in a way that electrical engineers take advantage of, when they design antennae). Things with **magnetic charge** g produce a magnetic field given by a Coulomb-like law:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r} \tag{11}$$

Something with magnetic charge g will feel a force in a magnetic field

$$\vec{F} = g\vec{B} \tag{12}$$

[1]In 1982, a Stanford research group led by Blas Cabrera Navarro made a detection of a single magnetic monopole, and then never detected another one ever again.

[2]. . . or something like that. . .

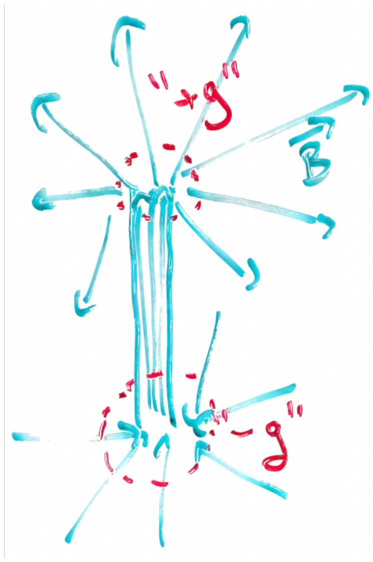

Figure 3: A solenoid imagined as two magnetic monopoles

and, in an electric field,

$$\vec{F} = -\frac{1}{c^2} g\vec{v} \times \vec{B} \tag{13}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light.

Most of these facts are curiosities, but can have some simple practical value. For example, an open-ended solenoid produces a magnetic field which naïvely seems like it requires careful accounting for nasty edge effects. In reality, as long as you are asking for the field a distance r much farther away from a solenoid than both its length L and radius R, this problem is very easy to solve using magnetic monopoles, as long as the solenoid is small ($r, z \gg L$) and "skinny" ($R \ll L$).

First, recall that Gauss's law for electric fields relates the electric flux Φ_e out of a closed surface is related to the enclosed charge Q_{enc} as

$$Q_{\text{enc}} = \epsilon_0 \Phi_e \tag{14}$$

In the presence of magnetic monopoles, the magnetic flux Φ_b behaves the same way with respect to the enclosed magnetic charge G_{enc}:

$$G_{\text{enc}} = \Phi_b / \mu_0 \tag{15}$$

The magnetic field produced inside of a solenoid is $B = \mu_0 NI/L$, where N is the number of coils. Therefore, the magnetic flux flowing in one end of the solenoid and out the other is

$$\Phi_b = \pi R^2 \mu_0 NI/L \tag{16}$$

If the solenoid is very skinny, then the magnetic field can be imagined to enter almost isotropically in one end and out the other. Thus, unless you look closely enough to resolve the solenoid, you just notice one spot where magnetic field flows into seemingly nowhere (with "magnetic charge" $-g$) and reappears out of a close-by, but slightly different location ($+g$). In this case, the magnetic charge would have to be

$$g = \Phi_b/\mu_0 = \pi R^2 NI/L \tag{17}$$

The magnetic field is just a superposition of these two monopoles:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{g}{(x^2+y^2+(z-L)^2)^{3/2}} (x\hat{x} + y\hat{y} + (z-L)\hat{z}) - \frac{\mu_0}{4\pi} \frac{g}{(x^2+y^2+z^2)^{3/2}} (x\hat{x} + y\hat{y} + z\hat{z}) \tag{18}$$

Since we only trust the result far away from the solenoid (so that generically $r \gg L$ and "probably" $z \gg L$), we can Taylor expand the first term (note that $r = \sqrt{x^2+y^2+z^2}$):

$$\frac{1}{(x^2+y^2+(z-L)^2)^{3/2}} \approx \frac{1}{r^3(1-2zL/r^2)^{3/2}} \approx \frac{1}{r^3}\left(1 + \frac{3zL}{r^2}\right) \tag{19}$$

Plugging this approximation in gives

$$\vec{B} \approx \frac{\mu_0 g}{4\pi r^3}\left[\left(1+\frac{3zL}{r^2}\right)(x\hat{x} + y\hat{y} + (z-L)\hat{z}) - (x\hat{x} + y\hat{y} + z\hat{z})\right] \tag{20}$$

or

$$\vec{B} \approx \frac{\mu_0 g L}{4\pi r^4}(3z\hat{r} - r\hat{z}) \tag{21}$$

Using $z = r\cos\theta$ gives

$$\vec{B} \approx \frac{\mu_0 g L}{4\pi r^3}(3\cos\theta\,\hat{r} - \hat{z}) = \frac{\mu_0 R^2 NI}{4r^3}(3\cos\theta\,\hat{r} - \hat{z}) \tag{22}$$

This can be compared to the electric field of an electric dipole with dipole moment $\vec{p} = qL\hat{z}$:

$$\vec{E} = \frac{qL}{4\pi\epsilon_0 r^3}(3\cos\theta\,\hat{r} - \hat{z}) \tag{23}$$