California Institute of Technology
Physics 1b Recitation
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"Out beyond ideas of wrongdoing and rightdoing, there is a field. I will meet you there." —Rumi

Contents

1 Maxwell's equations

Maxwell's equations govern the full behavior of the electric and magnetic fields, and are, in their integral form, given by

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = Q_{\mathrm{enc}}/\epsilon_0 \qquad \text{(Gauss's law)} \tag{1a}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0 \qquad \text{(no magnetic monopoles)} \tag{1b}$$

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \vec{B} \cdot \mathrm{d}\vec{A} \qquad \text{(Faraday's law)} \tag{1c}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\mathrm{enc}} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int \vec{E} \cdot \mathrm{d}\vec{A} \qquad \text{(Ampére's law)} \tag{1d}$$

We can condense these down into the following form:

$$\bigcirc\!\!\!\!\!\Phi_e = Q_{\mathrm{enc}}/\epsilon_0 \tag{2a}$$

$$\bigcirc\!\!\!\!\!\Phi_b = 0 \tag{2b}$$

$$\bigcirc\!\!\!\!\!\Gamma_e = -\dot{\Phi}_b \tag{2c}$$

$$\bigcirc\!\!\!\!\!\Gamma_b = \mu_0 I_{\mathrm{enc}} + \mu_0 \epsilon_0 \dot{\Phi}_e \tag{2d}$$

Note that this notation is a little non-standard, but concise.

Here, Φ and Γ denote **flux** and **circulation**, respectively, and subscript e and b denote that these quantities are meant to be computed with respect to the electric and magnetic fields, respectively. Circled quantities are *closed* integrals (i.e., closed surfaces for fluxes, closed loops for circulations). Dots denote time derivatives.

The terms on the right hand side are the **source terms**, and it is often helpful to think of them as "generating" the fields, in some sense. When presented with a problem to *find* a field (either $\vec{E}$ or $\vec{B}$) we may examine Equations 2 for the relevant sources. It is very helpful to take advantage of the symmetry which arises between the equations.

2 Practice problems

What follows are a series of practice problems which seem like they are very different from each other, but are unified in approach (i.e., applying Equations 2 for symmetry-informed surfaces/curves).

2.1 Spherical charge

Consider a static charge of *uniform* charge density ρ within a radius R (Figure 1).

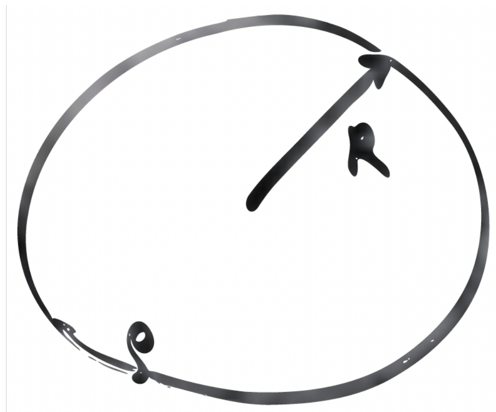

Figure 1: Sphere of charge

(a) Find a solution for the electric field everywhere.

We can draw a Gaussian sphere of radius r. It will enclose a charge $Q_{\text{enc}} = (4/3)\pi \text{max}(r, R)^3 \rho$.

Since the surface respects the symmetry of the system, the field is uniform across the sphere, and we thus have

$$\oint \Phi_e = 4\pi r^2 E = (4/3)\pi \max(r, R)^3 \rho/\epsilon_0 \tag{S1}$$

This yields

$$\vec{E} = E\hat{r} = \frac{\rho \max(r, R)^3}{3\epsilon_0 r^2} \tag{S2}$$

This is equivalent to

$$\vec{E} = \begin{cases} \frac{\rho}{3\epsilon_0} r\hat{r} & r < R \\ \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r} & r > R \end{cases} \tag{S3}$$

(b) Find a solution for the magnetic field everywhere.

We can use the analogous divergence law for magnetism,

$$\oint \Phi_b = 0 \tag{S4}$$

where now the right hand side is just zero.

Drawing the same Gaussian surface as before, we see that

$$\vec{B} = 0 \tag{S5}$$

2.2 Wires and solenoids

Figure 2: Wire carrying uniform current density

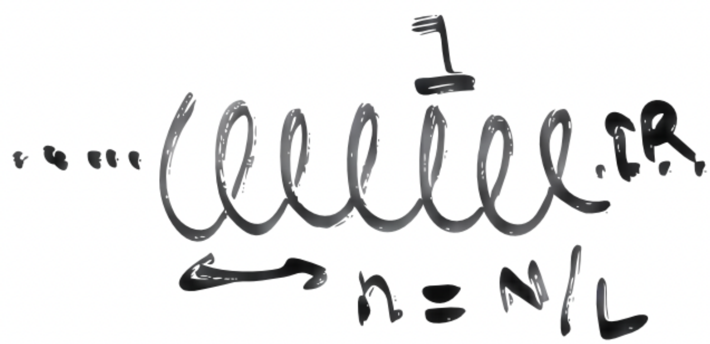

Figure 3: Solenoid

(a) Find a solution for the magnetic field everywhere due to a wire of steady, uniform current density J and radius R (Figure 2).

We can draw an Ampérian loop with radius r around the wire (this respects the problem symmetry). The circulation of the electric field is then

$$\textcircled{\Gamma_b} = 2\pi r B \tag{S6}$$

Ampére's law gives

$$\textcircled{\Gamma_b} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \dot{\Phi}_e \tag{S7}$$

Since there is no electric field, the second term vanishes. The first term is

$$I_{\text{enc}} = J\pi \max(r, R)^2 \tag{S8}$$

Then

$$\vec{B} = B\hat{\phi} = \frac{J \max(r, R)^2}{2r}\hat{\phi} \tag{S9}$$

or

$$\vec{B} = \begin{cases} \frac{\mu_0 J}{2} r\hat{\phi} & r < R \\ \frac{\mu_0 J R^2}{2r}\hat{\phi} & r > R \end{cases} \tag{S10}$$

(b) Find a solution for the magnetic field everywhere due to a solenoid of radius R carrying a current I, with n turns per unit length (Figure 3).

We can draw a rectangular Ampérian loop with length L along the direction of the solenoid, with one side in and the other out.

Far away, it would appear that there is no net current, and so one expects the magnetic field to vanish far away. By letting both sides of this rectangle be outside the solenoid, we see that it therefore must vanish everywhere outside of the solenoid.

The circulation when one of the sides is inside the solenoid is then just

$$\circled{\Gamma_b} = BL \tag{S11}$$

The enclosed current is now

$$I_{\text{enc}} = nIL \tag{S12}$$

Using Ampére's law, we have

$$\vec{B} = B\hat{z} = \mu_0 nI\hat{z} \tag{S13}$$

inside the solenoid, and $\vec{B} = 0$ outside.

(c) If the current I through the solenoid increases with time linearly as $I = Kt$, find a solution for the electric field everywhere.

Note that there is no net charge, so

$$\circled{\Phi_e} = 0 \tag{S14}$$

However, there *is* a net change in the magnetic flux. Drawing a circular Ampérian loop of radius r, the magnetic flux enclosed is

$$\Phi_b = B\pi \max(r, R)^2 = \mu_0 nKt\pi \max(r, R)^2 \tag{S15}$$

so that

$$\dot{\Phi}_b = \mu_0 nK\pi \max(r, R)^2 \tag{S16}$$

Then, using Faraday's law,

$$\circled{\Gamma_e} = 2\pi rE = -\mu_0 nK\pi \max(r, R)^2 \tag{S17}$$

so that

$$\vec{E} = -\frac{\mu_0 nK \max(r, R)^2}{2r}\hat{\phi} \tag{S18}$$

or

$$\vec{E} = \begin{cases} -\frac{\mu_0 nK}{2} r\hat{\phi} & r < R \\ -\frac{\mu_0 nKR^2}{2r}\hat{\phi} & r > R \end{cases} \tag{S19}$$

2.3 Plane charge

Consider a plane charge of uniform surface charge density σ.

Figure 4: Plane charge

(a) Find a solution for the electric field everywhere.

We can draw a Gaussian pillbox with area A, which encloses a charge $Q_{\text{enc}} = \sigma A$. The electric flux out of the pillbox is

$$\boxed{\Phi_e} = 2EA = \sigma A/\epsilon_0 \tag{S20}$$

so that

$$\vec{E} = \begin{cases} E\hat{z} & z > 0 \\ -E\hat{z} & z < 0 \end{cases} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{z} & z > 0 \\ -\frac{\sigma}{2\epsilon_0}\hat{z} & z < 0 \end{cases} \tag{S21}$$

(b) Now consider that $z < 0$ is a conductor, with $z = 0$ being the interface with the outside, carrying still a charge σ. If the charge density is made to increase linearly with time as $\sigma = Kt$, find a solution for the magnetic field for $z > 0$.

We can now pick an axis perpendicular to the plane, and draw an Ampérian circular loop with radius r around it. This loop will enclose an electric flux

$$\Phi_e = E\pi r^2 = Kt\pi r^2/2\epsilon_0 \tag{S22}$$

so that

$$\dot{\Phi}_e = K\pi r^2/2\epsilon_0 \tag{S23}$$

Since there are no currents passing through the loop, Ampére's law (with displacement current) just yields

$$\textcircled{\Gamma_e} = 2\pi r B = \mu_0\epsilon_0\dot{\Phi}_e = \mu_0 K\pi r^2/2 \tag{S24}$$

Therefore, we have

$$\vec{B} = \frac{1}{2}\mu_0 K\pi r^2\hat{\phi} \tag{S25}$$

Note that, because of the translational symmetry in this problem, we have found a solution of this situation, but not a unique one. This is also the case with every other problem in this collection, but it is very obvious here.

2.4 Induction

Consider a capacitor of capacitance C which is attached to itself, with area A. There is a linearly increasing magnetic field $\vec{B} = Kt$ passing perpendicularly through the wire loop.

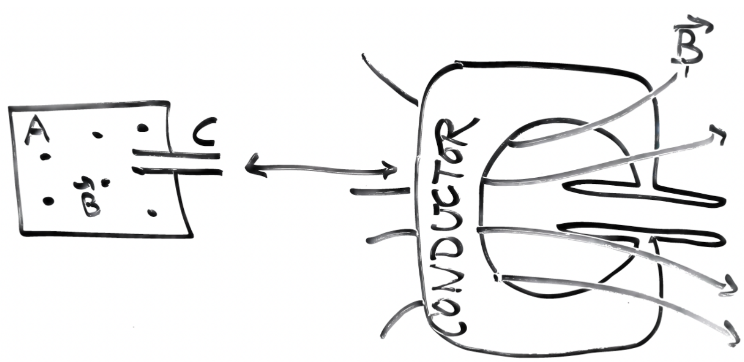

Figure 5: Self-attached capacitor threaded with a magnetic field

Find the charge stored on the capacitor, in steady state.

We can draw an Ampérian loop through the circuit. Because most of the circuit is a conductor, $\vec{E} = 0$ there. Therefore, the circulation must be contributed only by the electric field in the capacitor which we can say has a separation d (this is also the voltage V across the capacitor):

$$\textcircled{\Gamma_e} = Ed = V \tag{S26}$$

The magnetic flux through our loop is

$$\Phi_b = BA = KtA \tag{S27}$$

so that

$$\dot{\Phi}_b = KA \tag{S28}$$

Then Faraday's law gives

$$\textcircled{\Gamma_e} = V = -KA \tag{S29}$$

where the minus sign isn't particularly important (since the two plates will carry charges with different signs anyway). The charge here is then given by

$$|Q| = C|V| = CKA \tag{S30}$$