California Institute of Technology
Physics 101 Recitation
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Absolute Unit
10 April 2023

"My mathematics is simple: one plus one = one." —Dejan Stojanovic, The Shape

Contents

1 Practice Problems

1.1 The Long and Short of It

The answer to a physics problem can only depend on some scenario-specific parameters and constants (perhaps, but not necessarily, fundamental) which reflect the "importance" of some type of physics.

Table 1 lists a number of fundamental physics constants:

Constant	Value	When important?
speed of light c	$\sim 3 \times 10^8\,\mathrm{m\,s^{-1}}$	speeds are relativistic or electromagnetism is important
gravitational constant G	$\sim 7 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$	gravity is important
Planck's constant h	$7 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$	quantum mechanics is important

Table 1: Some fundamental physics concepts and when they are important.

In this Problem, we will determine which constants are most conveniently set to 1 (based on the relevant physics), and then guess 1 in those units to estimate the "natural" length scales desired.

(a) Estimate the radius of a "classical" gravitational orbit around a mass M which has an orbital frequency Ω.

> We set $G = M = \Omega = 1$. Then the radius a of an orbit is
>
> $$a \sim 1 \sim \sqrt[3]{\frac{GM}{\Omega^2}} \tag{S1}$$
>
> This is exactly correct.

(b) General relativity (Einstein's theory of gravity) extends special relativity (physics near the speed of light), and predicts that mass sufficiently compacted to form a black hole.

Estimate the radius of a black hole (called the **Schwarzschild radius**) with mass M.

> We set $G = c = M = 1$. Then the Schwarzschild radius R is
>
> $$R \sim 1 \sim \frac{GM}{c^2} \tag{S2}$$
>
> In reality, in these units, $R = 2$. For an Earth-mass black hole, $R \approx 1\,\mathrm{cm}$.

(c) In quantum mechanics, particles have wavelike properties. The wavelength of a particle is called its **de Broglie wavelength**, and defines the length scale at which quantum mechanics starts to become important for it.

Estimate the de Broglie wavelength of a particle with mass m and speed v.

We set $h = m = v = 1$. Then the de Broglie wavelength is

$$\lambda \sim 1 \sim \frac{h}{mv} \tag{S3}$$

This is exactly correct.

An electron moving at $0.01c$ (a typical speed in a hydrogen atom), this is $\lambda_c \approx 2.4 \times 10^{-10}\,\mathrm{m} = 2.4\,\text{Å}$.

(d) Estimate the length scale at which a particle of mass m must be treated both quantum mechanically and relativistically.

This is called the **Compton wavelength**.

We set $h = c = m = 1$. Then the Compton wavelength is

$$\lambda \sim 1 \sim \frac{h}{mc} \tag{S4}$$

For an electron, $\lambda \approx 2.4 \times 10^{-12}\,\mathrm{m} = 2.4\,\mathrm{pm}$.

(e) Estimate the length scale at which both general relativity and quantum mechanics are important.

This is called the "Planck length."

We set $G = c = \hbar = 1$. Then the Planck length is given by ℓ_p is

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}\,\mathrm{m} \tag{S5}$$

1.2 May You Live in Turbulent Times

Suppose that energy is injected into a fluid of large scale, at a rate $\dot{\epsilon}$ per unit mass. That energy will "cascade" into smaller and smaller scales until it can be dissipated by viscosity. Assume that the fluid is incompressible.

(a) In the "inertial range" (scales below the energy injection scale), how does the wavenumber spectrum $\mathrm{d}\epsilon/\mathrm{d}k$ depend on k?

The wavenumber spectrum has units of $[\mathrm{d}\epsilon/\mathrm{d}k] = [\epsilon]/[k] = [L]^3/[T]^2$ (note that $[\epsilon] = [L]^2/[T]^2$). The energy injection rate has units of $[\dot{\epsilon}] = [\epsilon]/[T] = [L]^2/[T]^3$, and the wavenumber has units $[k] = [L]^{-1}$.

The only "independent" way to arrange these dimensionful quantities into a dimensionless quantity is:

$$\frac{k^5(\mathrm{d}\epsilon/\mathrm{d}k)^3}{\dot{\epsilon}^2} = \text{const.} \tag{S6}$$

We see that

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}k} \propto \dot{\epsilon}^{2/3}k^{-5/3} \tag{S7}$$

This $-5/3$ exponent is the famous exponent within **Kolmogorov turbulence**.

(b) What is the length scale at which viscosity (parameterized by the kinematic viscosity ν, with units $[\nu] = [L]^2/[T]$) can dissipate the turbulent energy?

The only way to combine ν, $\dot{\epsilon}$, and k into a dimensionless quantity is

$$\frac{k^4\nu^3}{\dot{\epsilon}} = \text{const.} \tag{S8}$$

Then we see that the dissipative (**Kolmogorov**) length scale $\ell \sim 1/k$ is

$$\ell \propto \left(\frac{\nu^3}{\dot{\epsilon}}\right)^{1/4} \tag{S9}$$

where the prefactor is of order unity.

(c) Explain why computing the properties of **magnetohydrodynamic turbulence** (magnetic turbulence in a highly conductive medium) would be so hard.

The addition of another dimensionful quantity associated with the magnetic field (e.g., the Alfvén velocity v_A) causes dimensional analysis to stop working.

This means that at least one (perhaps very sketchy) *physical* argument is required to justify why some variables should occur in certain combinations. Also, because the magnetic field defines a special direction, this is complicated even more by there being multiple wavenumbers of relevance (since this could differ based on whether the direction is parallel or perpendicular to the magnetic field).

Alexander Schekochichin has a very amusing review on this topic[a].

[a] https://arxiv.org/abs/2010.00699

1.3 Blast to the Gas

The shock front of an explosion at a given moment has a radius R and velocity v, and is slowed down as it slams into the surrounding gas, which has a density ρ.

(a) Using R, v, and ρ, estimate the energy E of the explosion.

> The only way to construct an energy E from R, v, and ρ is
>
> $$E \sim \rho R^3 v^2 \tag{S10}$$

(b) Figure 1 shows the shock wave produced by the first-ever detonation of a nuclear weapon in history, at a time $t \approx 15\,\mathrm{ms}$ after the explosion.

Estimating $v \sim R/t$, estimate the energy E of the explosion in kilotons. Note that the surrounding air has a density $\rho \approx 1\,\mathrm{kg\,m^{-3}}$, and that a kiloton (of TNT) is a unit of energy equal to 4.2×10^{12} J.

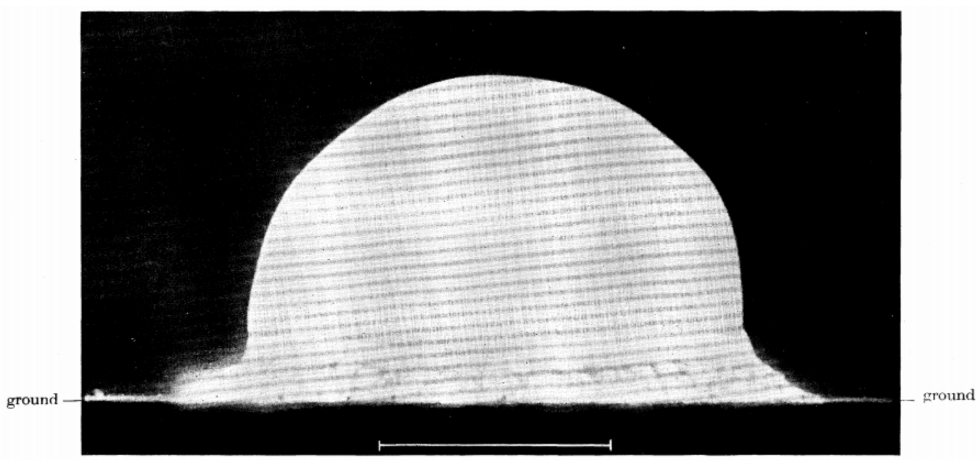

FIGURE 7. The ball of fire at $t = 15$ msec., showing the sharpness of its edge.

Figure 1: Snapshot of the Trinity nuclear test explosion at $t = 15\,\mathrm{ms}$

Estimating R by eye from Figure 1, we approximate

$$v \sim \frac{R}{t} \sim \frac{100\,\mathrm{m}}{15\,\mathrm{ms}} \sim 6700\,\mathrm{m\,s^{-1}} \tag{S11}$$

Then

$$E \sim 4.1 \times 10^{13}\,\mathrm{J} \sim 11\,\mathrm{kt} \tag{S12}$$

The real answer has been estimated to be $\approx 25\,\mathrm{kt}$. Enrico Fermi was famously able to estimate the yield of the explosion by dropping a piece of paper while watching it from afar and observing how far it was blown. The physicist G. I. Taylor was later able to estimate the yield of the bomb using some publicly released images (including the one shown in Figure 1) before the yield was declassified by the United States government.

(c) How does R scale with t?

Taking $v \sim R/t$, we have

$$E \sim \rho R^5/t^2 \tag{S13}$$

so that

$$R \propto t^{2/5} \tag{S14}$$

(d) Suppose that, instead of a constant amount of energy (injected into the explosion from the beginning), there is a steady energy deposition rate $\dot{E}$.

How does R evolve with time?

Now, we can take $E = \dot{E}t$, so that

$$\dot{E}t \sim \rho R^5/t^2 \tag{S15}$$

This yields

$$R \propto t^{3/5} \tag{S16}$$