California Institute of Technology
Physics 101 Recitation **Breaking Point**
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"It's the children the world almost breaks who grow up to save it." —Frank Warren

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(a) If an object has a density ρ, elastic modulus E, and breaking strain ϵ^*, at what maximum speed v can it collide with a hard surface without breaking?

The kinetic energy of a moving object is $(1/2)mv^2 = (1/2)\rho V v^2$, and must fully be absorbed by the elastic energy $(1/2)EV\epsilon^2$:

$$\frac{1}{2}\rho V v^2 \simeq \frac{1}{2}EV\epsilon^2 \tag{S1}$$

At the breaking stress,

$$v^2 \simeq (E/\rho)(\epsilon^*)^2 \tag{S2}$$

To make this easier to interpret, define the sound speed:

$$c_s^2 \simeq E/\rho \tag{S3}$$

Then

$$v \simeq c_s\epsilon^* \tag{S4}$$

(b) From what maximum height h can a person jump from without fracturing their bones, ignoring air resistance?

Water has a bulk modulus $E \simeq 2\,\mathrm{GPa}$. *Bone consists of a thin, hard outer layer called "cortical bone" with* $E \simeq 17\,\mathrm{GPa}$ *and* $\epsilon^* \approx 0.02$ *surrounding spongy "trabecular bone" with* $E \simeq 0.4\,\mathrm{GPa}$ *and* $\epsilon^* \approx 0.5$*, where* E *refers to the compressive elastic modulus.*[1] *Cortical bone makes up* $d \simeq 5\,\mathrm{mm}$ *of the outer layer of the femur*[2]*.*

[1] https://www.ncbi.nlm.nih.gov/pmc/articles/PMC5601257/

[2] http://vetmed.agriculturejournals.cz/artkey/vet-201307-0001_a-comparison-of-the-microarchitecture-of-lower-limb-long-bones-between-some-animal-models-and-humans-a-review.php

The speed v will be related to the height h by

$$v = \sqrt{2gh} \tag{S5}$$

so that

$$2gh \simeq c_s^2(\epsilon^*)^2 \tag{S6}$$

where ϵ^* will be limited by the most brittle element in the body (the cortical bone). Then

$$h \simeq c_s^2(\epsilon^*)^2/2g \tag{S7}$$

All the bulk moduli which are listed are of similar orders of magnitude: muscle is probably relatively close to water. To be more careful, we may notice that E serves the role of a spring constant, and that spring constants of springs in parallel are additive.

Let us assume that the person is attempting to land on their feet: the relative proportions are likely to be somewhat similar to that of a typical cross-sectional cut of a person in any direction (except for very specific cross-sections with no trabecular bone, although these are likely dominated by muscle instead).

We have

$$E_{\text{eff}}A_{\text{tot}} \simeq E_{\text{muscle}}A_{\text{muscle}} + E_{\text{cortical}}A_{\text{cortical}} + E_{\text{trabecular}}A_{\text{trabecular}} \tag{S8}$$

For my leg at the thinnest point, $R_{\text{leg}} \simeq 3\,\text{cm}$ and $R_{\text{bone}} \simeq 1\,\text{cm}$, and

$$A_{\text{muscle}} \simeq \pi(R_{\text{leg}}^2 - R_{\text{bone}}^2) \simeq 25\,\text{cm}^2 \tag{S9a}$$

$$A_{\text{cortical}} \simeq 2\pi R_{\text{bone}} d \simeq 3\,\text{cm}^2 \tag{S9b}$$

$$A_{\text{trabecular}} \simeq \pi R_{\text{bone}}^2 \simeq 3\,\text{cm}^2 \tag{S9c}$$

Then

$$E_{\text{eff}} \simeq 3\,\text{GPa} \tag{S10}$$

and

$$c_s \simeq \sqrt{E_{\text{eff}}/\rho} \simeq 1.7\,\text{km}\,\text{s}^{-1} \tag{S11}$$

using water density.

Then

$$h \simeq c_s^2(\epsilon^*)^2/2g \simeq 60\,\text{m} \tag{S12}$$

This seems to be roughly consistent with the "death zone" to order-of-magnitude[a]. This number seems on the higher end of the estimate for survival heights, but people do survive higher heights than this sometimes (though typically with injuries).

[a] https://www.forbes.com/sites/stevensalzberg/2012/05/06/falling-300-feet-onto-a-rock-and-surviving-to-tell-the-tale/?sh=43e27c044f2dhttps://www.forbes.com/sites/stevensalzberg/2012/05/06/falling-300-feet-onto-a-rock-and-surviving-to-tell-the-tale/?sh=43e27c044f2d

(c) Now consider air resistance. How small does an animal have to be before it is "immune to fall damage?"

For high Reynolds numbers (which animals typically are), we can find the terminal velocity by finding the speed at which the drag force balances out gravity:

$$\frac{1}{2}C_D\rho_{\mathrm{air}}Av^2 \approx mg \tag{S13}$$

so that

$$v \simeq \sqrt{2L\rho_{\mathrm{water}}g/C_D\rho_{\mathrm{air}}} \tag{S14}$$

The threshold animal size L, assuming a similar morphology and composition as a person, is given by

$$2L\rho_{\mathrm{water}}g/C_D\rho_{\mathrm{air}} \simeq c_s^2(\epsilon^*)^2 \tag{S15}$$

or

$$L \simeq \frac{C_D}{2}\frac{c_s^2(\epsilon^*)^2}{g}\frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{water}}} \simeq 6\,\mathrm{cm} \tag{S16}$$

Mice are on the threshold of this limit.

(d) Estimate the minimum surface area a parachute has to have in order to save a person.

The chief purpose of a parachute is to decrease the terminal velocity by increasing the "effective area" of the person.

We see from part (c) that the terminal velocity becomes

$$v \simeq \sqrt{2mg/C_D\rho_{\rm air}A} \tag{S17}$$

Then

$$2mg/C_D\rho_{\rm air}A = c_s^2(\epsilon^*)^2 \tag{S18}$$

Rearranging, we have

$$A \simeq \frac{2mg}{c_s^2(\epsilon^*)^2C_D\rho_{\rm air}} \simeq 1\,{\rm m}^2 \tag{S19}$$

where note that m is the total mass of the person plus parachute. Because a person should be able to wear a parachute as a backpack, this is likely to be a small (or at least order-unity) fraction of a person's mass.

Note that this is quite a bit smaller than normal parachutes, although based on our assumptions this is not necessarily mysterious. We have essentially enforced a threshold on the velocity of $v \simeq c_s\epsilon^* \simeq 30\,{\rm m\,s^{-1}}$, which is likely to be on the same order of magnitude as (but on the very upper limit of) the maximum survival speed for a human, based on *fracture*. Certainly things like bruising, etc., can occur after collisions at much lower speeds, and the scaling of area with this speed tolerance is $A \propto v^2$.

A factor of 3 difference in the tolerable speed accounts for a factor of 10 increase in the tolerable A. In one news story, a person was able to make a skydive using a parachute with $A \simeq 4\,{\rm m}^{-2}$[a].

[a] https://www.hindustantimes.com/world/man-jumps-14-000ft-with-smallest-parachute-sets-world-record/story-mEi82PR9kWw5joanTMbgEO.html

(e) We have trouble tracking space debris larger than $R_{\rm debris} \gtrsim 5\,{\rm cm}$[3].

Estimate the radius of a hole this would make in a solid block of steel (yield stress $\sigma \simeq 250\,{\rm GPa}$) on the same orbit as the International Space Station.

What if the debris is the size of a grain of sand ($\simeq 1\,{\rm mm}$)?

[3] https://www.nbcnews.com/news/all/astronaut-photographs-cosmic-bullet-hole-space-station-flna6c9738512

The space debris is likely rock, and would therefore have a density $\rho \simeq 3\,\mathrm{g\,cm^{-3}}$.

The orbital speed at the surface of the Earth is

$$v \simeq \sqrt{GM_\oplus/R_\oplus} \simeq 8\,\mathrm{km\,s^{-1}} \tag{S20}$$

We can then balance the energy contributed by the moving particle to the energy of the fractured steel:

$$\frac{1}{2}mv^2 \simeq \frac{2}{3}\pi R^3\sigma \tag{S21}$$

Then

$$R \simeq \left(\frac{3mv^2}{4\pi\sigma}\right)^{1/3} \simeq \left(\frac{\rho_{\mathrm{debris}}v^2}{\sigma}\right)^{1/3} R_{\mathrm{debris}} \simeq 10R_{\mathrm{debris}} \tag{S22}$$

where $\rho_{\mathrm{debris}} \simeq 3\,\mathrm{g\,cm^{-3}}$.

Thus, a piece of space debris with $R_{\mathrm{debris}} \simeq 5\,\mathrm{cm}$ should be able to make a fracture of size $R \simeq 50\,\mathrm{cm}$, and a large piece of sand with $R_{\mathrm{debris}} \simeq 1\,\mathrm{mm}$ should be able to make a hole $R \simeq 1\,\mathrm{cm}$.

Note that some of the energy may be dissipated as heat, waves, or go into the kinetic energy of unbound material, so these are likely to be upper estimates.

The expression obtained here resembles (with a different power) the result of some previous work which considers rods instead of spherical craters[a].

[a] https://www.sciencedirect.com/science/article/pii/S0734743X16304614