California Institute of Technology
Physics 101 Recitation
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Diffuse the Situation
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"When you take breaks, your diffuse mode is still working away in the background." —Barbara Oakley, A Mind for Numbers: How to Excel at Math and Science

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1 Practice Problem: *Stumbling into the Answer*

(a) In the Sun, photons are constantly scattered by processes such as electron scattering, which occurs with the Thomson cross section:

$$\sigma_T \approx 0.7\,\mathrm{b} = 7 \times 10^{-29}\,\mathrm{m}^2 \tag{1}$$

Roughly estimate how long it takes a photon at the center of the Sun to reach the surface. Note that the density at the center at the Sun is $\simeq 10^2$ times denser than the mean density of the entire Sun.

Compare this to how long it would take if there were no collisions.

The average density of the Sun is

$$\rho \simeq \frac{M_\odot}{4\pi R_\odot^3/3} \simeq \frac{M_\odot}{4R_\odot^3} \tag{S1}$$

Then the number density of electrons is approximately (by charge neutrality)

$$n \simeq \frac{fM_\odot}{4m_p R_\odot^3} \tag{S2}$$

where $f \simeq 10^2$.

Then the mean free path is

$$\lambda = \frac{1}{n\sigma} = \frac{4m_p R_\odot^3}{fM_\odot \sigma_T} \simeq 1.7 f^{-1}\,\mathrm{cm} \tag{S3}$$

For uncorrelated steps, the mean variances of the position add:

$$\langle r^2 \rangle = n\lambda^2 \tag{S4}$$

Setting $r = R_\odot$, the number of collisions that a photon would undergo before having a large chance of diffusing out of the star is

$$n \simeq R_\odot^2/\lambda^2 \tag{S5}$$

This would take a time

$$T \simeq n\lambda/c \simeq R_\odot^2/\lambda c \approx 3 \times 10^6\,\mathrm{yr} \tag{S6}$$

Note that there is quite a bit of variation in the answers here depending on what is adopted for f, since this can dramatically increase the number of collisions a photon experiences and thus greatly increase the time required to escape.

In any case, this is much, *much* larger (by a factor $R_\odot/\lambda$)than the escape time of a hypothetical photon which would not need to collide with anything:

$$T_0 = R_\odot/c \approx 2\,\mathrm{s} \tag{S7}$$

(b) Roughly estimate how long it would take a given air molecule you breathe out to diffuse to the other side of the Earth, *ignoring* advection.

The number density of air molecules is

$$n \simeq \frac{\rho}{28 m_p} \approx 2 \times 10^{-25}\,\mathrm{m}^{-3} \tag{S8}$$

using $\rho \approx 1\,\mathrm{kg\,m}^{-3}$.

The cross section of two air molecules interacting is probably $\sigma = \pi r^2$ where $r \simeq 3\,\text{Å}$, yielding

$$\lambda = \frac{1}{n\sigma} \simeq 200\,\mathrm{nm} \tag{S9}$$

To diffuse around the Earth (ignoring geometric effects of the Earth's curvature), one must diffuse a distance $D \simeq \pi R_\oplus$. Then the number of steps needed to reach this distance is

$$\langle r^2 \rangle \simeq \pi^2 R_\oplus^2 = n\lambda^2 \tag{S10}$$

so that

$$n \simeq \frac{\pi^2 R_\oplus^2}{\lambda^2} \simeq 10^{28} \tag{S11}$$

The typical velocity of a given air particle is

$$v \simeq \sqrt{\frac{kT}{28 m_p}} \approx 300\,\mathrm{m\,s}^{-1} \tag{S12}$$

Then the total time this would take is

$$T = n\lambda/v \approx 2\,\mathrm{Gyr} \tag{S13}$$

(c) Jet streams have typical speeds $v \simeq 200\,\mathrm{km\,hr}^{-1}$. Roughly estimate how long would it take an molecule to cross from one side of the Earth to the other, if it hitched a ride on a jet stream?

We simply have

$$T \simeq \frac{\pi R_\oplus}{v} \approx 4\,\mathrm{d} \tag{S14}$$

This is much, *much* smaller than the analogous diffusion time found in part (b).

(d) Consider measuring n variables x_n, each introducing an error σ. Estimate the *error* on the sum if the errors are

(i) statistical.

Statistical errors behave as random walks:

$$\sigma_{\text{tot}}^2 = n\sigma^2 \tag{S15}$$

Then

$$\sigma_{\text{tot}} = \sqrt{n}\sigma \tag{S16}$$

(ii) systematic.

Because the errors add "coherently," the total error is

$$\sigma_{\text{tot}} = n\sigma \tag{S17}$$

If n is large, this answer would deviate a *lot* from the analogous statistical error-based quantity.

2 Practice Problem: *Be a Conservationist*

Conservation laws take the form

$$\frac{\partial q}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{2}$$

where q is the density of the conserved quantity and $\vec{j}$ is the current, and can be any vector field.

The fact that this is a conservation law can be seen by integrating over some volume v and applying Gauss's law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v q\,\mathrm{d}V = -\int_{\partial v} \vec{j}\cdot \mathrm{d}\vec{A} \tag{3}$$

(a) Let the temperature T obey the diffusion equation,

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{4}$$

Given the density ρ and specific heat capacity C, find an expression for the heat flux.

The thermal energy density is given by

$$\varepsilon_T = \rho C T \tag{S18}$$

Multiplying Equation 4 by ρC, we have

$$\frac{\partial(\rho C T)}{\partial t} = \nabla \cdot (\rho C \alpha \nabla T) \tag{S19}$$

Rearranging and substituting, we have

$$\frac{\partial \varepsilon_T}{\partial t} + \nabla \cdot \vec{q} = 0 \tag{S20}$$

where the heat flux $\vec{q}$ is given by

$$\vec{q} = -\rho C \alpha \nabla T \tag{S21}$$

(b) The free Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi \tag{5}$$

Show that ψ is a conserved density, and find its current.

The Schrödinger equation is just a complex diffusion equation, and can be rewritten as

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \left(\frac{\hbar}{2mi} \nabla \psi \right) = 0 \tag{S22}$$

This is a conservation law for ψ, with current

$$\vec{j} = \frac{\hbar}{2mi} \nabla \psi \tag{S23}$$

(c) Under a potential $V = V(\vec{x})$, the Schrödinger becomes

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \tag{6}$$

Show that $\psi^* \psi$ is still a conserved density, and find its current.

The trick from before doesn't work anymore. However, we may multiply the Schrödinger equation (Equation 6) by ψ^*:

$$i\hbar\psi^*\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi \tag{S24}$$

We can take the complex conjugate of this:

$$-i\hbar\psi\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi^*\psi \tag{S25}$$

We can subtract these equations to eliminate V:

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = i\hbar\frac{\partial(\psi^*\psi)}{\partial t} = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) \tag{S26}$$

Note that

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* = \psi^*\nabla^2\psi + \nabla\psi^*\cdot\nabla\psi - \nabla\psi\cdot\nabla\psi^* - \psi\nabla^2\psi^* = \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) \tag{S27}$$

Then

$$\frac{\partial(\psi^*\psi)}{\partial t} + \nabla\cdot\left(\frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\right) = 0 \tag{S28}$$

The probability current is then

$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{S29}$$

(d) We can modify the conservation law to

$$\frac{\partial q}{\partial t} + \nabla\cdot\vec{j} = F \tag{7}$$

Write this equation in integral form. What is the effect of F?

The integral form of Equation 7 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v q\,\mathrm{dV} = -\int_{\partial v}\vec{j}\cdot\mathrm{d}\vec{A} + \int_v F\,\mathrm{d}V \tag{S30}$$

We see that the F term is a source term which adds material independently of it entering and exiting the boundaries of some domain.